Course Info

MA13 - Multivariable Calculus	Fall 2009
Meeting Times	MWF 10:00-10:50pm Th 9:00-9:50am
Location	Seeley Mudd 206
Instructor	Ben Hutz
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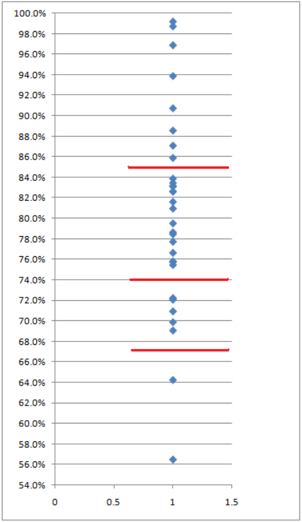
Office	Seeley Mudd 502
Office Hours	M 2-3pm, T 5-7pm, Th 10-11am
Text	Multivariable Calculus 6rd Edition, James Stewart

Overview/Evaluation

Elementary vector calculus; introduction to partial derivatives; extrema of functions of several variables; multiple integrals in two and three dimensions; line integrals in the plane; Green's theorem.

- 5% Group Projects
- 15% Homework
- 50% Three in-class exams
- 30% Final Exam

Curving Grades: Final grades from a previous course



Getting Help

- The Moss Quantitative Center provides math help. It is located in 202 Merrill Science Center and you can find the math hours at `http://www.amherst.edu/~qcenter/`.
- The Dean of Students Office can arrange for peer tutoring. Information can be found at `http://www.amherst.edu/~dos/acadsupport.html`.
- Please come see me during my office hours! If you have a conflict and cannot make my office hours, please email me and we can set up an appointment for another time.

Three Dimensions

What separates multivariable from single variable calculus is the addition of a third dimension. You have a 3-dimensional coordinate axes labeled x, y, z which are broken up into

- 8 octants
- 3 coordinate planes

In 2-dim you have

1. points
2. lines

Two distinct lines either intersect in a point or are parallel. In 3dim you have

1. points
2. lines
3. planes

Two distinct planes either intersect in a line or are parallel. Two distinct lines intersect in a point, are parallel, or are skew.

The distance between two points

You can have *distance formula* formula

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

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$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Proof.

Rectangular box with P_1 and P_2 in opposite corners and label the in-between corners as A, B. So you have from Pythagorean Theorem that

$$|P_1P_2|^2 = |P_1B|^2 + |P_2B|^2$$
$$|P_1B|^2 = |P_1A|^2 + |AB|^2$$

Combining these we get

$$|P_1P_2|^2 = |P_1A|^2 + |AB|^2 + |P_2B|^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2$$

A problem

Problem

Find an equation for describing the set of all points equidistant from $(-1, 2, 3)$ and $(3, 0, -1)$. (What common geometric object is the set?)

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Proof.

We need

$$\sqrt{(x+1)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x-3)^2 + y^2 + (z+1)^2}$$

which is

$$2x - y - 2z + 1 = 0.$$

This is a plane. $\square$

Another example

Problem

Find an equation of the sphere with center $(2, -6, 4)$ and radius 5.
Describe its intersection with each of the coordinate planes.

Another example

Problem

Find an equation of the sphere with center $(2, -6, 4)$ and radius 5. Describe its intersection with each of the coordinate planes.

Proof.

The equation is $(x-2)^2 + (y+6)^2 + (z-4)^2 = 25$. The intersection with the xy-plane($z = 0$) is $(x-2)^2 + (y+6)^2 = 9$ a circle of radius 3 centered at $(2, -6)$. The intersection with the xz-plane ($y = 0$ is $(x-2)^2 + (z-4)^2 = -9$. In other words, there intersection is empty. The intersection with the yz-plane($x = 0$) is $(y+6)^2 + (z-4)^4 = 21$. In other words, a circle of radius $\sqrt{21}$ centered at $(-6, 4)$. $\square$

Vectors

Definition

A *vector* has both a magnitude a direction. A *scalar* has only a magnitude. For example, 20mph is a scalar, but 20mph due east is a vector.

Some Terminology:

- Initial Point
- Terminal Point
- Components
- Magnitude or length

We will denote a vector as $\overrightarrow{a} = \langle a_1, a_2, a_3 \rangle$ where a_1 is the number of units to move in the x direction, etc.

Magnitude and Unit vectors

Definition

The *magnitude* of a vector $\overrightarrow{a}$ is the distance from its initial point to its terminal point and is denoted $|a|$. (The initial point is assumed to be the origin unless otherwise specified).

Example

$|\langle 3,4\rangle| = \sqrt{3^3+4^2} = 5.$

Definition

A *unit vector* is a vector with magnitude 1 and is generally thought of as a “direction”.

Example

$\frac{1}{\sqrt{3}}\langle 1,1,1\rangle.$

Parallel in 3-dimensions

Problem

How do we tell if two lines are parallel?

Parallel in 3-dimensions

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How do we tell if two lines are parallel?

Proof.

In two dimensions we have $y = mx + b$. If $m_1 = m_2$ then they are parallel, in other words if $\langle m1, 1 \rangle = c \langle m2, 1 \rangle$ for some constant c. Then we can easily generalize to 3-dimensions with vectors: two lines are parallel if their direction vectors are equal up to scaling. $\square$

Vector Addition and Scalar Multiplication

- Addition of vectors $\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle$. (triangle and parallelogram laws(commutativity))
- scalar multiplication $c \langle a_1, a_2 \rangle = \langle ca_1, ca_2 \rangle$.
- There are at least two ways to define vector multiplication (dot product, cross product) which we will discuss later.

Alternate Notation

We define

$$\hat{i} = \langle 1, 0, 0 \rangle \quad \hat{j} = \langle 0, 1, 0 \rangle \quad \hat{k} = \langle 0, 0, 1 \rangle$$

and so we can write

$$\langle a_1, a_2, a_3 \rangle = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}.$$

Position and Displacement Vectors

Definition

Given a point $P = (P_1, P_2, P_3)$ its *displacement vector* is the vector with initial point the origin and terminal point P and is denoted $\overrightarrow{P}$.

Definition

The *displacement vector* from P to Q is the vector

$$\overrightarrow{PQ} = \overrightarrow{Q} - \overrightarrow{P}$$

and is not the same as $\overrightarrow{QP}$.

Dot Product

Definition

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Example

$\langle 1, 2, 3 \rangle \cdot \langle -1, 0, 5 \rangle = -1 + 0 + 15 = 14.$

Notice two things

1. the result is a scalar.
2. it is possible to get 0 without one of the vectors being $\vec{0}$.

Properties

Proposition

The dot product satisfies the following properties.

1. *commutative* $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
2. $\vec{a} \cdot \vec{a} = \left|\vec{a}\right|^2$
3. $c\vec{a} \cdot \vec{b} = \vec{a} \cdot c\vec{b} = c(\vec{a} \cdot \vec{b})$
4. $\vec{a} \cdot \vec{b} = \left|\vec{a}\right| \left|\vec{b}\right| \cos\theta$

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3. $c\vec{a} \cdot \vec{b} = \vec{a} \cdot c\vec{b} = c(\vec{a} \cdot \vec{b})$
4. $\vec{a} \cdot \vec{b} = \left|\vec{a}\right| \left|\vec{b}\right| \cos\theta$

Proof.

1. $a_1 b_1 + a_2 b_2 + a_3 b_3 = b_1 a_1 + b_2 a_2 + b_3 a_3$.
2. $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 = (\sqrt{a_1^2 + a_2^2 + a_3^3})^2 = \left|\vec{a}\right|^2$.

$\square$

Proof Continued

Proof.

3

$$\begin{aligned} c\vec{a} \cdot \vec{b} &= \langle ca_1, ca_2 \rangle \cdot \langle b_1, b_2 \rangle = ca_1 b_1 + ca_2 b_2 \\ &= a_1 cb_1 + a_2 cb_2 = \vec{a} \cdot c\vec{b} \\ &= ca_1 b_1 + ca_2 b_2 = c(a_1 b_1 + a_2 b_2) = c(\vec{a} \cdot \vec{b}) \end{aligned}$$

Proof Continued

Proof.

3.
$$\begin{aligned} c\vec{a}\cdot\vec{b} &= \langle ca_1, ca_2\rangle \cdot \langle b_1, b_2\rangle = ca_1b_1 + ca_2b_2 \\ &= a_1cb_1 + a_2cb_2 = \vec{a}\cdot c\vec{b} \\ &= ca_1b_1 + ca_2b_2 = c(a_1b_1 + a_2b_2) = c(\vec{a}\cdot\vec{b}) \end{aligned}$$

4. Label three sides of a triangle $\vec{a}, \vec{b}, \vec{a-b}$. Then the law of cosines says that
$$\left|\vec{a}-\vec{b}\right|^2 = |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2\,|\vec{a}|\,\left|\vec{b}\right|\cos\theta. \tag{1}$$

□

Proof Continued

Proof.

Writing

$$\left|\vec{a}-\vec{b}\right|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}$$
$$= |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2\vec{a}\cdot\vec{b}.$$

So the equation (1) becomes

$$-2\vec{a}\cdot\vec{b} = -2\,|\vec{a}|\,\left|\vec{b}\right|\cos\theta.$$

$\square$

A simple example

Problem

Find the angle between $\langle 1, 1 \rangle, \langle 0, 1 \rangle$.

A simple example

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Find the angle between $\langle 1, 1\rangle, \langle 0, 1\rangle$.

Proof.

This is $\cos\theta = \frac{1}{\sqrt{2}}$ which is $\frac{\pi}{4}$. $\square$

Orthogonality

Corollary

Two non-zero vectors are perpendicular if and only if $\vec{a} \cdot \vec{b} = 0$.

Orthogonality

Corollary

Two non-zero vectors are perpendicular if and only if $\vec{a} \cdot \vec{b} = 0$.

Proof.

Property 4 states that $\vec{a} \cdot \vec{b} = |\vec{a}| \left|\vec{b}\right| \cos\theta$. If $\vec{a}$ and $\vec{b}$ are orthogonal then $\theta = \pi/2$ and $\cos\theta = 0$.

If $\vec{a} \cdot \vec{b} = 0$, then $|\vec{a}| \left|\vec{b}\right| \cos\theta = 0$. Since the vectors are nonzero we must have $\cos\theta = 0$ and hence $\theta = \pi/2$. □

Example

Problem

1. Show that $2\hat{i} + 2\hat{j} - \hat{k}$ is perpendicular to $5\hat{i} - 4\hat{j} + 2\hat{k}$
2. Find a vector perpendicular to $\langle 1, -1, 2 \rangle$. How many are there?

Example

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Proof.

1. $\langle 2, 2, -1\rangle \cdot \langle 5, -4, 2\rangle = 10 - 8 - 2 = 0.$
2. It is clear that $\langle 1, -1, 2\rangle \cdot \langle -1, 1, 0\rangle = 0$, but in fact there are infinitely many choices. We just need $\langle x, y, z\rangle \langle 1, -1, 2\rangle = x - y + 2z = 0$ (and this is a plane).

□

Another Example

Problem

Find the angle between the diagonal of a cube and one of its edges.

Another Example

Problem

Find the angle between the diagonal of a cube and one of its edges.

Proof.

We want the angle between $\langle 1, 1, 1\rangle$ and $\langle 1, 0, 0\rangle$. That angle satisfies

$$\cos\theta = \frac{1}{\sqrt{3}}$$

So we have

$$\theta = \arccos\frac{1}{\sqrt{3}}.$$

$\square$

Projections

Definition

1. $\text{comp}_a b = \frac{a \cdot b}{|a|}$ (scalar projection)
2. $\text{proj}_a b = \frac{a \cdot b}{|a|} \frac{a}{|a|}$. (vector projection)

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Problem

Find the vector projection of $\vec{a} = \langle 1, 2, 4 \rangle$ on $\vec{b} = \langle 4, -2, 4 \rangle$.

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1. $\mathrm{comp}_a b = \frac{a \cdot b}{|a|}$ (scalar projection)
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Find the vector projection of $\vec{a} = \langle 1, 2, 4 \rangle$ on $\vec{b} = \langle 4, -2, 4 \rangle$.

Proof.

the vector projection is $\frac{a \cdot b}{|b|^2} b$. $\vec{a} \cdot \vec{b} = 4 - 4 + 16 = 16$ and $\vec{b} \cdot \vec{b} = 16 + 4 + 16 = 36$. So we have $\frac{16}{36} \vec{b} = \langle \frac{16}{9}, \frac{-8}{9}, \frac{16}{9} \rangle$. □

Perpendicular Distances

Problem

Find the distance from $(0, 0)$ to $y = -2x + 1$.

Perpendicular Distances

Problem

Find the distance from $(0,0)$ to $y = -2x + 1$.

Proof.

Pick any point on the line, say $(0,1)$, we need to project $\langle 0,1\rangle$ onto the perpendicular $\langle 2,1\rangle$. This is

$$\text{comp}_a b = \frac{1}{\sqrt{5}}$$

$\square$

Perpendicular Distances

Problem

1. Use scalar projection to show that the distance from a point (x_1, y_1) to a line $ax + by + c = 0$ is given by
$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$
2. Find the distance from $(4, -1)$ to $3x - 4y + 5 = 0$.

Remark

There is a similar formula for planes in 3-dim, but we need to learn about planes first...

Perpendicular Distances

Proof.

1. The slope of the line is $\langle -b, a\rangle$, so a vector perpendicular to the line is $\langle a, b\rangle$. Choose any point (x_2, y_2) on the line. Then the distance from (x_1, y_1) to the line is given by

$$d = \mathrm{comp}_a b = \frac{\langle x_1 - x_2, y_1 - y_2\rangle \cdot \langle a, b\rangle}{\sqrt{a^2+b^2}}$$
$$= \frac{ax_1 + by_1 - ax_2 - by_2}{\sqrt{a^2+b^2}} = \frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}.$$

2. So we have

$$d = \frac{12+4+5}{\sqrt{25}} = \frac{21}{5}.$$

□

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Given a line defined by two points P_1 and P_2, find the distance to the point Q.

Perpendicular Distances

Problem

Given a line defined by two points P_1 and P_2, find the distance to the point Q.

Proof.

We first take the $\vec{a} = \text{proj}_{\vec{P_1P_2}} \vec{QP_2}$. Then the perpendicular distance is given by

$$\left| \vec{QP_2} - \vec{a} \right| .$$

□

Perpendicular Distances

Problem

Given a line defined by the points $(0, 0, 0)$ and $(1, 1, 1)$, find the distance to the point $(2, 0, 0)$.

Perpendicular Distances

Problem

Given a line defined by the points $(0,0,0)$ and $(1,1,1)$, find the distance to the point $(2,0,0)$.

Proof.

We have

$$\vec{P_1P_2} = \langle 1,1,1 \rangle \quad \text{and} \quad \langle -1,1,1 \rangle .$$

We compute

$$\vec{a} = \text{proj}_{\vec{P_1P_2}} \vec{QP_2} = \text{proj}_{\langle 1,1,1 \rangle} \langle -1,1,1 \rangle = \frac{1}{\sqrt{3}} \frac{\langle 1,1,1 \rangle}{\sqrt{3}} = \frac{1}{3} \langle 1,1,1 \rangle .$$

Then the perpendicular distance is given by

$$\left| \vec{QP_2} - \vec{a} \right| = \left| \langle -1,1,1 \rangle - \frac{1}{3} \langle 1,1,1 \rangle \right| = \left| \left\langle -\frac{4}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle \right| = \frac{2}{3}\sqrt{6}.$$

Cross Product

We now define a second type of vector product

Definition

$$\vec{a} \times \vec{b} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$
$$= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Example

Problem

Compute $\langle 2, -1, 3\rangle \times \langle 0, 2, 1\rangle$.

Example

Problem

Compute $\langle 2, -1, 3\rangle \times \langle 0, 2, 1\rangle$.

Proof.

We have

$$\begin{aligned}
\langle 2, -1, 3\rangle \times \langle 0, 2, 1\rangle &= \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 0 & 2 & 1 \end{pmatrix} \\
&= -\hat{i} + 0 + 4\hat{k} - 0 - 6\hat{i} - 2\hat{j} \\
&= \langle -7, -2, 4\rangle .
\end{aligned}$$

$\square$

Basic Arithmetic

Theorem

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $c\vec{a} \times \vec{b} = \vec{a} \times c\vec{b} = c(\vec{a} \times \vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$. *(scalar triple product)*
5. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ *(vector triple product)*

Proof.

Write them out... $\square$

Orthogonality

Theorem

$\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

Proof.

We check

$$\begin{aligned}\vec{a} \cdot (\vec{a} \times \vec{b}) &= \langle a_1, a_2, a_3 \rangle \cdot \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \\ &= a_1 a_2 b_3 - a_1 a_3 b_2 + a_2 a_3 b_1 - a_2 a_1 b_3 + a_3 a_1 b_2 - a_3 a_2 b_1 = 0.\end{aligned}$$

Similarly for $\vec{b} \cdot (\vec{a} \times \vec{b})$ $\square$

Example

Problem

Find a unit vector orthogonal to both $\langle 1, 3, 2 \rangle$ and $\langle -1, 4, 3 \rangle$.

Example

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Find a unit vector orthogonal to both $\langle 1, 3, 2\rangle$ and $\langle -1, 4, 3\rangle$.

Proof.

A vector orthogonal to 2 given vectors is the cross product of those two vectors. So we have

$$\det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 2 \\ -1 & 4 & 3 \end{pmatrix} = \langle 1, -5, 7\rangle .$$

Making this a unit vector yields

$$\frac{1}{\sqrt{75}}\langle 1, -5, 7\rangle .$$

□

Alternate Form

Theorem

$|a \times b| = |a|\,|b| \sin\theta.$

Proof.

$$\begin{aligned} |a \times b|^2 &= |a|^2\,|b|^2 - (a \cdot b)^2 \quad \text{(by direct comp.)} \\ &= |a|^2\,|b|^2 (1 - \cos^2\theta) \\ &= |a|^2\,|b|^2 \sin^2\theta \end{aligned}$$

So we have the desired equality since the squares are all squares of positive numbers since $\sin\theta$ is positive for $0 < \theta < \pi$. $\square$

Corollaries

Corollary

Two non-zero vectors $\vec{a}$ *and* $\vec{b}$ *are parallel if and only if* $\vec{a} \times \vec{b} = 0$.

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Proof.

If they are parallel then $\theta = 0$ and hence $\sin\theta = 0$.
If $\vec{a} \times \vec{b} = |\vec{a}| \left|\vec{b}\right| \sin\theta = 0$ and $\vec{a}, \vec{b}$ are nonzero by assumption, then we must have $\sin\theta = 0$ and hence $\theta = 0$. $\square$

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Corollary

$\left|\vec{a} \times \vec{b}\right|$ is the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

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Corollary

$\left|\vec{a} \times \vec{b}\right|$ is the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

Proof.

$\left|\vec{b}\right| \sin\theta$ is the height and $|\vec{a}|$ is the length. □

Volume

Corollary

The magnitude of the scalar triple product, $V = \left|\vec{a} \cdot (\vec{b} \times \vec{c})\right|$*, is the volume of the parallelepiped determined by the vectors* $\vec{a}, \vec{b}$*, and* $\vec{c}$*.*

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Proof.

The area of the base is $\left|\vec{b} \times \vec{c}\right|$ and if θ is the angle between $\vec{a}$ and $\vec{b} \times \vec{c}$, then $\left|\vec{a}\right| |\cos\theta|$ is the height. (we use $|\cos|$ in case $\theta > \pi/2$). $\square$

Summary of Additional Properties

Theorem

1. $\vec{a} \times \vec{b}$ *is perpendicular to both* $\vec{a}$ *and* $\vec{b}$*.*
2. $|a \times b| = |a|\,|b| \sin \theta$*.*
3. *Two non-zero vectors* $\vec{a}$ *and* $\vec{b}$ *are parallel if and only if* $\vec{a} \times \vec{b} = 0$*.*
4. $\left|\vec{a} \times \vec{b}\right|$ *is the area of the parallelogram determined by* $\vec{a}$ *and* $\vec{b}$*.*
5. *The magnitude of the scalar triple product,* $V = \left|\vec{a} \cdot (\vec{b} \times \vec{c})\right|$*, is the volume of the parallelepiped determined by the vectors* $\vec{a}$*,* $\vec{b}$*, and* $\vec{c}$*.*

Example

Problem

Given the three points

$$P_1 : (1, 1, -1) \quad P_2 : (2, 4, 0) \quad P_3 : (0, 2, -2)$$

Find the area of the triangle $P_1 P_2 P_3$.

Example

Problem

Given the three points

$$P_1 : (1, 1, -1) \quad P_2 : (2, 4, 0) \quad P_3 : (0, 2, -2)$$

Find the area of the triangle $P_1 P_2 P_3$.

Proof.

Two sides of the triangle are given by the vectors $\vec{P_1P_2} = \langle 1, 3, 1\rangle$ and $\vec{P_1P_3} = \langle -1, 1, -1\rangle$. The magnitude of the cross product is the area of the parallelogram, which is twice the area of the desired triangle. So we compute

$$\langle 1, 3, 1\rangle \times \langle -1, 1, -1\rangle = \langle -4, 0, 4\rangle$$

which has magnitude $4\sqrt{2}$. So the area of the triangle is $2\sqrt{2}$. $\square$

Basic Information for Lines

The minimum information to construct a line:

1. 2 points
2. a point and a direction.

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Definition

$$r_0 + t\vec{v} \quad \text{for } t \in \mathbb{R}$$

where r_0 is the position vector of any point on the line and $\vec{v}$ is the direction (slope) of the line.

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Lines can be

- Intersecting
- Parallel
- Skew.

Equation of a Line

Definition

Parametric Equations

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

Definition

Symmetric Equations

$$\frac{x - x_0}{a} = \frac{y = y_0}{b} = \frac{z - z_0}{c}$$

Examples

Problem

Find a point on the line and a direction vector.

1. $\langle 3+t, 1-t, 3+2t\rangle$.
2. $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z-3}{2}$.

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Proof.

1. Goes through the point $(3, 1, 3)$ with direction $\frac{1}{\sqrt{6}} \langle 1, -1, 2 \rangle$.
2. Goes through the point $(1, -1, 3)$ with direction $\frac{1}{\sqrt{17}} \langle 2, -3, 2 \rangle$.

$\square$

Parallel Lines

Problem

Find the symmetric and parametric equations for the line through the point $(2, 5, 3)$ that is parallel to the line

$$\frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}.$$

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$$\frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}.$$

Proof.

The line must be in the direction of $\langle -4, -3, 2\rangle$ so we have the parametric equations

$$\langle 2-4t, 5-3t, 3+2t\rangle$$

and the symmetric equations

$$\frac{x-2}{-4} = \frac{y-5}{-3} = \frac{z-3}{2t}.$$

Intersection Points

To determine if two lines intersect, we must determine if their equations have a common solution.

Problem

Determine if these two lines intersect:

$$L1 : \langle 3 + 2t, -2 + 4t, -1 - 3t \rangle$$
$$L2 : \langle -1 + 3s, -2 + 2s, -2 - s \rangle$$

Solution

Proof.

We use the z coordinate to solve $-1 - 3t = -2 - s$ and so $s = -1 + 3t$. Substituting into ont of the other two equations yields

$$3 + 2t = -1 + 3(-1 + 3t)$$

which is $7t = 7$ which is $t = 1$ and so $s = 2$. So we have the solution $(5, 2, -4)$ for both lines. $\square$

Skew Lines

Problem

Determine if these two lines intersect:

$$L1 : \langle -2 + t, 4 - 3t, -5 - t \rangle$$
$$L2 : \langle 3 - s, 5 + 2s, 4 + s \rangle$$

Solution

Proof.

Attempting to solve we find from the x coordinates that $t = 5 - s$. Substituting into one of the other two equations yields

$$4 - (5 - s) = 5 + 2s$$

and so $s = -6$ and then $t = 11$. So we have the point $(9, -29, -16)$ and $(9, -7, -2)$. So the lines do not intersection since the solution is inconsistent. $\square$

Another Example

Problem

For the three following lines determine for the three possible pairs of two lines whether they are parallel, intersecting, or skew. If intersecting, determine the point of intersection.

$$L_1 : -6t\hat{i} + (1+9t)\hat{j} - 3t\hat{k}$$
$$L_2 : x = 5 + 2t, y = -2 - 3t, z = 3 + t$$
$$L_3 : \frac{x-3}{2} = \frac{-1-y}{5} = z - 2$$

Solution

Proof.

Lines L_1, L_2 are parallel since their directions $\langle -6, 9, -3\rangle$ and $\langle 2, -3, 1\rangle$ are parallel (they differ by the multiple -3).
Lines L_2, L_3 intersect at $(1, 4, 1)$.
Lines L_1, L_3 have direction $\langle -6, 9, -3\rangle$ and $\langle 2, -5, 1\rangle$ respectively, so are clearly not parallel. If they intersect we must have

$$-6t = 2s + 3$$
$$-3t = 2 + s$$

So we must have

$$2s + 3 = 2s + 2$$

which is not possible. So the lines do not intersect. Therefore, they must be skew. □

Planes

Minimum information needed to define a plane

1. a point and a (normal) vector
2. Two non-parallel vectors
3. 3 (non-colinear) points

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To determine the equation for a plane consider that the normal vector is perpendicular to every vector in the plane. If P_0 is represented by the position vector $\vec{r_0}$ and P is any other point, represented by the vector $\vec{r}$, then the plane is determined buy

$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$$

where $\vec{n}$ is the normal vector. In particular we have

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$$

In particular we can represent any plane as

$$ax + by + cz + d = 0.$$

A Simple Example

Problem

Find the plane through $(1, 1, 1)$ with normal $\vec{n} = \langle 0, 2, -1 \rangle$.

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Proof.

This is the plane

$$0(x-1) + 2(y-1) - (z-1) = 0$$

which is

$$2y - z - 1 = 0.$$

□

Two vectors

Problem

Find the plane containing the vectors $\vec{a} = \langle 2, -4, 4 \rangle$ and $\vec{b} = \langle 4, -1, -2 \rangle$ and the point $P = (1, 3, 2)$.

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Proof.

We compute $\vec{n} = \vec{a} \times \vec{b} = \langle 12, 20, 14 \rangle$ and given $P = (1, 3, 2)$ on the plane we have

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

which is

$$6x + 10y + 7z = 50.$$

$\square$