# Math 12 Spring 2009: Exam 2

**Name:**

**Instructions:** There are 4 questions on this exam each of which is scored out of 8 points for a total of 32 points. You may not use any outside materials (eg. notes or books). You may use your calculator **ONLY** for problem 1. You have 50 minutes to complete this exam. Remember to fully justify your answers.

**Score:**

**Problem 1.** Consider $\ln 2 = \int_1^2 \frac{1}{x} dx$. Use **only one of** midpoint rule, trapezoidal rule, or Simpson's rule to answer the following two questions.

(a) How many terms are needed to approximate this integral to within $\frac{1}{250}$?

(b) Find the value to within $\frac{1}{250}$.

*Proof.* We have

$$f''(x) = \frac{2}{x^3} \quad \text{and} \quad f^{(4)}(x) = \frac{24}{x^5}.$$

**Midpoint:**

(a) We have $EM(n) \leq \frac{K(a-b)^3}{24n^2}$, we can choose $K = 2$ and so we need $n$ such that

$$\frac{2}{24n^2} = \frac{1}{12n^2} < \frac{1}{250}.$$

So we need $n \geq 5$.

(b) We have

$$\int_1^2 \frac{1}{x} dx \approx \frac{1}{5} \left( f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9) \right) \approx 0.6919$$

**Trapezoidal:**

(a) We have $ET(n) \leq \frac{K(a-b)^3}{12n^2}$, we can choose $K = 2$ and so we need $n$ such that

$$\frac{2}{12n^2} = \frac{1}{6n^2} < \frac{1}{250}.$$

So we need $n \geq 7$.

(b) We have

$$\int_1^2 \frac{1}{x} dx \approx \frac{1}{14} \left( f(1) + 2f(8/7) + 2f(9/7) + 2f(10/7) + 2f(11/7) + 2f(12/7) + 2f(13/7) + f(2) \right)$$
$$\approx 0.6944$$

**Simpson's:**

(a) We have $ES(n) \leq \frac{K(b-a)^5}{180n^4}$, we can choose $K = 24$ and so we need $n$ such that

$$\frac{24}{180n^4} = \frac{2}{15n^4} < \frac{1}{250}.$$

So we need $n \geq 3$.

(b) Since $n$ must be even for Simpson's Rule we use $n = 4$. We have

$$\int_1^2 \frac{1}{x} dx \approx \frac{1}{12} \left( f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2) \right) \approx 0.6920$$

□

**Problem 2.** For the following two improper integrals. Determine whether they converge or diverge. If they converge find their value.

(a) $\int_0^1 \frac{x^3}{\sqrt{1-x^2}}dx$

(b) $\int_1^\infty \frac{1}{x-\ln x}dx$.

*Proof.*

(a) We get

$$\begin{aligned}
\lim_{c\to 1^-}\int_0^c \frac{x^3}{\sqrt{1-x^2}}dx &= \lim_{c\to 1^-}\int_{x=0}^{x=c} \frac{\sin^3\theta\cos\theta}{\sqrt{\cos^2\theta}}d\theta \\
&= \lim_{c\to 1^-}\int_{x=0}^{x=c} \sin^3\theta d\theta \\
&= \lim_{c\to 1^-}\int_{x=0}^{x=c} \sin\theta - \sin\theta\cos^2\theta d\theta \\
&= \lim_{c\to 1^-}[-\cos\theta + \frac{\cos^3\theta}{3}]_{x=0}^{x=c}.
\end{aligned}$$

Drawing the triangle,we get

$$\begin{aligned}
&= \lim_{c\to 1^-}[-\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2}]_0^c \\
&= \lim_{c\to 1^-}[-\sqrt{1-c^2} + \frac{1}{3}(1-c^2)^{3/2} + 1 - \frac{1}{3}] = \frac{2}{3}.
\end{aligned}$$

Note, that it is also possible to do this integral by making the substitution $u = 1-x^2$ to get

$$\begin{aligned}
\lim_{c\to 1^-}\int_0^c \frac{x^3}{\sqrt{1-x^2}}dx &= \lim_{c\to 1^-}\int_{x=0}^{x=c} -\frac{1}{2}\frac{1-u^2}{\sqrt{u}}du \\
&= \lim_{c\to 1^-} -\frac{1}{2}\int_{x=0}^{x=c} u^{-1/2} - u^{1/2}du \\
&= \lim_{c\to 1^-} -\frac{1}{2}\left[2u^{1/2} - \frac{2}{3}u^{3/2}\right]_{x=0}^{x=c} \\
&= \lim_{c\to 1^-}\left[-(1-x^2)^{1/2} + \frac{1}{3}(1-x^2)^{3/2}\right]_0^c = \frac{2}{3}.
\end{aligned}$$

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(b) Since $x > \ln x \geq 0$ for $1 \leq x \leq \infty$, we have $\frac{1}{x-\ln x} \geq \frac{1}{x}$.

$$\begin{aligned}
\int_1^\infty \frac{1}{x} &= \lim_{c\to\infty}\int_1^c \frac{1}{x}dx \\
&= \lim_{c\to\infty}\ln c = \infty.
\end{aligned}$$

so the smaller integral diverges, so the integral diverges.

□

**Problem 3.** Find the arc length of the part of the curve $y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$ from $x = 1$ to $x = 2$.

*Proof.* We have

$$y' = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$$

and so

$$1 + (y')^2 = \frac{1}{4}x^4 + \frac{1}{4}x^{-4} + \frac{1}{2} = \frac{1}{4}(x^2 + x^{-2})^2$$

and so

$$\begin{aligned} s &= \int_1^2 \frac{1}{2}(x^2 + x^{-2})dx \\ &= \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^2 \\ &= \frac{8}{6} - \frac{1}{4} - \frac{1}{6} + \frac{1}{2} \\ &= \frac{7}{6} + \frac{1}{4} = \frac{34}{24} = \frac{17}{12}. \end{aligned}$$

□

**Problem 4.** Find the volume when the region bounded by $y = 2x^4$ and $y = x^2 + 1$ is rotated about the $x$-axis.

*Proof.* You can find the intersection points either by noting that $(-1, 2)$ and $(1, 2)$ are point of intersection or solving $2x^4 = x^2 + 1$ with the quadratic equation to get

$$x^2 = \frac{1 \pm \sqrt{1+8}}{4}.$$

So we have $x^2 = 1$ or $x^2 = -\frac{1}{2}$. So we must have $x^2 = 1$ and hence $x = \pm 1$.

Using the washer method we have

$$\begin{aligned} V &= \int_{-1}^{1} \pi((x^2+1)^2 - (2x^4)^2)dx \\ &= \pi \int_{-1}^{2} x^4 + 2x^2 + 1 - 4x^8 dx \\ &= \pi \left( \frac{x^5}{5} + \frac{2}{3}x^3 + x - \frac{4}{9}x^9 \right)_{-1}^{1} \\ &= 2\pi \left( \frac{1}{5} + \frac{2}{3} + 1 - \frac{4}{9} \right) \\ &= 2\pi \frac{64}{45} = \frac{128\pi}{45}. \end{aligned}$$

□