Math 13 Fall 2009: Exam 2

November 4, 2009

Name:

Instructions: There are 4 questions on this exam each scored out of 8 points for a total of 32 points. You may not use any outside materials(eg. notes or calculators). You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Problem 1. Define

$$f(x,y) = \begin{cases} \frac{2x^4 - x^3 + xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

(a) Determine where $f(x,y)$ is continuous.

(b) Use the limit definition of partial derivatives to compute $f_x(0,0)$.

Proof.

(a) The function $f(x,y)$ is a rational function so is continuous everywhere its denominator is defined. Its denominator is the sum of two squares, so is 0 only when $x = y = 0$. So we just need to determine whether $f(x,y)$ is continuous at $(0,0)$. So we just need to check whether $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$. We first see

$$\begin{aligned} \left|\frac{2x^4 - x^3 + xy^2}{x^2+y^2}\right| &= \left|\frac{2x^4 - x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2}\right| \\ &\leq \left|\frac{2x^4 - x^3}{x^2} + \frac{xy^2}{y^2}\right| \\ &\leq \left|\frac{2x^4 - x^3}{x^2}\right| + \left|\frac{xy^2}{y^2}\right| \\ &= \left|2x^2 - x\right| + |x| \end{aligned}$$

So in particular

$$-(\left|2x^2 - x\right| + |x|) \leq \frac{2x^4 - x^3 + xy^2}{x^2+y^2} \leq \left|2x^2 - x\right| + |x|.$$

Taking the limits of the two outside functions we have

$$\begin{aligned} \lim_{x\to 0} -(\left|2x^2 - x\right| + |x|) &= 0 \\ \lim_{x\to 0} \left|2x^2 - x\right| + |x| &= 0 \end{aligned}$$

so by the squeeze theorem we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0).$$

Therefore $f(x,y)$ is continuous on $\mathbb{R}^2$.

(b) We compute

$$\begin{aligned} f_x(0,0)) &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{2h^4 - h^3}{h^2} - 0}{h} \\ &= \lim_{h\to 0} \frac{2h^2 - h}{h} \\ &= \lim_{h\to 0} 2h - 1 \\ &= -1. \end{aligned}$$

□

Problem 2.

(a) Given $z = 2x^2 - 3xy + 7y^2$ and $x = u \sin v, y = v \cos u$, find $\frac{\partial z}{\partial u}$ in terms of u and v.

(b) Given $f(x, y, z) = \sqrt{xyz}$ and two points $P = (2, 1, 2)$ and $Q = (-1, 1, 6)$. Find the directional derivative of f at P in the direction of Q.

Proof.

(a) We have

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ &= (4x - 3y)\sin v + (-3x + 14y)(-v \sin u) \\ &= 4u \sin^2 v - 3v \cos u \sin v + 3uv \sin v \sin u - v^2 \cos u \sin u. \end{aligned}$$

(b) We have $\nabla f = \left\langle \frac{yz}{2\sqrt{xyz}}, \frac{xz}{2\sqrt{xyz}}, \frac{xy}{2\sqrt{xyz}} \right\rangle$ and

$$\nabla f(2, 1, 2) = \frac{1}{2}\langle 1, 2, 1 \rangle.$$

The direction is given by

$$\vec{u} = \frac{\langle -3, 0, 4 \rangle}{5}.$$

So we compute

$$D_u f = \frac{-3 + 0 + 4}{10} = \frac{1}{10}.$$

□

Problem 3. Find the maximal volume of a rectangular box which has three faces in the coordinates planes and one vertex in the first octant on the paraboloid $z = 4 - x^2 - y^2$.

Proof. Let (x, y, z) be the corner of the box on the paraboloid. Then we need to maximize $V = xyz$ subject to $z + x^2 + y^2 - 4 = 0$. So we have the Lagrange system

$$\begin{aligned} yz &= \lambda 2x \\ xz &= \lambda 2y \\ xy &= \lambda \\ 0 &= z + x^2 + y^2 - 4 \end{aligned}$$

So we have

$$\begin{aligned} yz &= 2x^2 y \\ xz &= 2xy^2 \\ 0 &= z + x^2 + y^2 - 4 \end{aligned}$$

We have $\lambda = xy$ and $x, y \neq 0$ since otherwise the volume is 0, so

$$\begin{aligned} z &= 2x^2 \\ z &= 2y^2 \\ 0 &= z + x^2 + y^2 - 4 \end{aligned}$$

This gives $x^2 = y^2$ and hence $x = \pm y$. From the constraint we have

$$2y^2 + 2y^2 = 4$$

which is

$$y = \pm 1.$$

and so we have the possible solutions $(x, y) = (\pm 1, \pm 1)$. Since we must be in the first octant, we can only have $(1, 1)$. Therefore, we have $x = 1, y = 1, z = 2$, for a maximal volume of 2. $\square$

Problem 4. Classify the critical points of $f(x, y) = 6xy^2 - 2x^3 - 3y^4$.

Proof. We find the critical points as

$$f_x = 6y^2 - 6x^2 = 0$$
$$f_y = 12xy - 12y^3 = 0.$$

If $y = 0$, then we have $x = 0$ and the critical point $(0, 0)$.

If $y \neq 0$ we have

$$f_x : x^2 = y^2$$
$$f_y : x = y^2.$$

so we have the critical points $(1, \pm 1)$.

Applying the second derivative test we compute

$$f_{xx} = -12x$$
$$f_{xy} = 12y$$
$$f_{yy} = 12x - 36y^2$$

So we have

$$D(0, 0) = 0$$
$$D(1, 1) = (-12)(-24) - (12)^2 > 0$$
$$D(1, -1) = (-12)(-24) - (-12)^2 > 0$$

At $(1, \pm 1)$ we have a local maximum since $D > 0$ and $f_{xx} < 0$ at those points. However, at $(0, 0)$ the test is inconclusive. Examining the function we see that

$$f(0, y) = -3y^4$$
$$f(x, 0) = -3x^3$$

For $(0, 0)$ we have $f(0, 0) = 0$. Near $(0, 0)$, for $x = 0$ and $y \neq 0$ we get a positive value. For $y = 0$ and $x < 0$ we get a positive value. These values both exist arbitrarily close to $(0, 0)$ so we know that $(0, 0)$ is neither a max nor a min and is therefore a saddle point. □