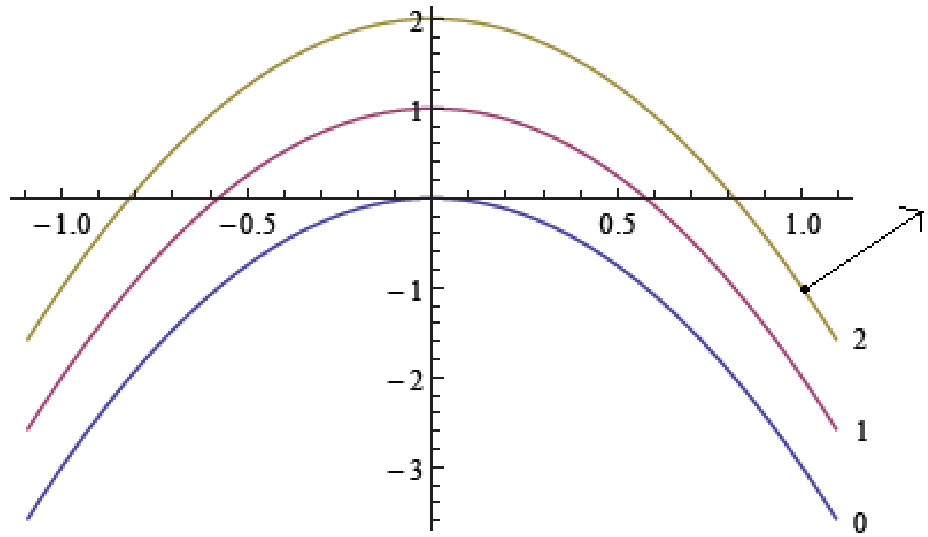
Math 211, Multivariable Calculus, Fall 2011
Midterm II Practice Exam 1 Solutions

1. *A cannon is fired on the Moon at an angle of* 30 *degrees above the horizontal. At what speed should the cannon be fired to hit a target that is* 100 *meters away (and on the same level)? You can assume that the acceleration due to gravity on the Moon is* 1.6 *meters per second squared.*

   The acceleration of the cannonball will be given by
   $$\mathbf{a}(t) = \langle 0, -1.6 \rangle.$$
   The initial velocity (if the initial speed is v_0) is
   $$\mathbf{v}(0) = \langle v_0 \cos(\pi/6), v_0 \sin(\pi/6) \rangle$$
   so the velocity at time t will be
   $$\mathbf{v}(t) = \langle v_0 \sqrt{3}/2, v_0/2 - 1.6t \rangle.$$
   If the cannon is placed at the origin so the initial position of the cannonball is $(0,0)$, then the position at time t is
   $$\mathbf{r}(t) = \langle v_0 t \sqrt{3}/2, v_0 t/2 - 0.8t^2 \rangle.$$
   Now if we want to hit the target at position $(100,0)$, we would need to have
   $$100 = v_0 t \sqrt{3}/2$$
   and
   $$0 = v_0 t/2 - 0.8t^2$$
   for some particular value of t. The first equation gives us
   $$t = \frac{200}{v_0 \sqrt{3}}$$
   and so the second gives
   $$t = \frac{v_0}{1.6}.$$
   Therefore we need
   $$v_0^2 \sqrt{3} = 320$$
   and so
   $$v_0 = \sqrt{320/\sqrt{3}}.$$
   Thus the cannon should be fired at this speed (in meters per second).

2. *Sketch a diagram of the k-level curves (for $k = 0, 1, 2$) for the function*
   $$f(x,y) = 3x^2 + y.$$
   Show on your graph the direction of the gradient vector of f at the point $(1,-1)$.



3. *Let* $\mathbf{r}(t)$ *be a parametrization of the level curve of the function* $f(x, y)$ *that passes through* (a, b). *Suppose that both* f *and* $\mathbf{r}$ *are differentiable everywhere. Prove that the gradient vector* $\nabla f(a, b)$ *is perpendicular to the level curve at* (a, b).

   Since $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ is a level curve of f, we have

   $$f(\mathbf{r}(t)) = c$$

   for all t, where c is some constant. Differentiating this equation with respect to t, and using the chain rule, we get

   $$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0.$$

   If $\mathbf{r}(t_0) = (a, b)$, then evaluating at t_0 we get

   $$\frac{\partial f}{\partial x}(a, b)\frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(a, b)\frac{dy}{dt}(t_0) = 0.$$

   But this can be rewritten as

   $$\nabla f(a, b) \cdot \mathbf{r}'(t_0) = 0.$$

   This tells us that the gradient vector $\nabla f(a, b)$ is perpendicular to $\mathbf{r}'(t_0)$ which is the tangent vector to the level curve at the point (a, b). Thus $\nabla f(a, b)$ is perpendicular to the level curve.

4. *Find the linear approximation to the function*

   $$f(x, y) = e^{2x}(\sin(3y) + \cos(3y))$$

   at the point $(0, 0)$ *and use your answer to estimate* $f(0.001, -0.002)$.

   The partial derivatives of f are:

   $$\frac{\partial f}{\partial x} = 2e^{2x}(\sin(3y) + \cos(3y))$$

and

$$\frac{\partial f}{\partial y} = 3e^{2x}(\cos(3y) - \sin(3y)).$$

At the point $(0,0)$ we have

$$\frac{\partial f}{\partial x}(0,0) = 2, \frac{\partial f}{\partial y}(0,0) = 3.$$

Therefore the linear approximation is

$$l(x,y) = 1 + 2(x-0) + 3(y-0) = 1 + 2x + 3y.$$

Therefore

$$f(0.001, -0.002) \approx 1 + 2(0.001) + 3(-0.002) = 0.996.$$

5. *Show that the function*

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0); \end{cases}$$

is not differentiable at $(0,0)$ but is differentiable at all other points (x,y). (Substantial partial credit is available for this problem if you approach it the right way, even if you do not completely finish it.)

Firstly, at points other than $(0,0)$, we have

$$\frac{\partial f}{\partial x} = \frac{(x^2+y^2)3x^2 - x^3(2x)}{(x^2+y^2)^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{-2yx^3}{(x^2+y^2)^2}.$$

Both of these are defined at every point other than $(0,0)$ and are continuous wherever they are defined. Therefore, since the partial derivatives are continuous at and around every point other than $(0,0)$, the function f is differentiable at every point other than $(0,0)$.

To see that f is not differentiable at $(0,0)$ we have to show that the relevant limit either does not exist or is not equal to 0. First, we need to find the partial derivatives at $(0,0)$. We have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

Therefore, the linear approximation, if it exists, at $(0,0)$ is given by

$$l(x,y) = 0 + 1(x-0) + 0(y-0) = x.$$

To see if f is differentiable, we have to look at

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-l(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\frac{x^3}{x^2+y^2}-x}{\sqrt{x^2+y^2}}.$$

Simplifying this limit, we get

$$\lim_{(x,y)\to(0,0)} \frac{-xy^2}{(x^2+y^2)^{3/2}}.$$

To show this limit is not zero, suppose we approach $(0,0)$ along the line $y=x$. Along this line, the function is given by

$$f(x,y) = f(x,x) = \frac{-x^3}{(2x^2)^{3/2}} = \frac{-1}{2^{3/2}} \neq 0.$$

Thus the overall limit is not zero and so f is not differentiable at $(0,0)$.