Amherst College
Department of Economics
Economics 58
Fall 2005

Name_Solutions______

Second Examination

(Theory of the Firm)

There are three questions on this seventy-five minute examination. The first question is worth 40 points. Questions 2 and 3 are each worth 30 points.

1. Suppose that a firm has a production function given by $q = k^{0.5} + l^{0.5}$. The remainder of this problem uses this production function to study various aspects of this firm's behavior.

a. What sort of returns to scale does this production function exhibit?

$f(2k, 2l) = (2k)^{.5} + (2l)^{.5} = 2^{.5} f(k,l) < 2f(k,l)$ so the function exhibits diminishing returns to scale.

b. Show how you would use the principle of cost-minimization to derive the cost function for this firm. You should show all of your work that permits you to arrive at the final form: $C(q,v,w) = q^2\left(\frac{vw}{v+w}\right)$.

$\mathcal{L} = vk + wl + \lambda(q - k^{.5} - l^{.5})$ First order conditions for a constrained minimum imply

$$\frac{l^{.5}}{k^{.5}} = \frac{v}{w} \quad or \quad l = \frac{v^2}{w^2}k \quad hence \quad q = \frac{v}{w}k^{.5} + k^{.5} = \frac{v+w}{w}k^{.5}$$

$$Hence \quad k = q^2\left(\frac{w}{w+v}\right)^2 \quad similarly \quad l = q^2\left(\frac{v}{w+v}\right)^2$$

Total costs are therefore $C = vk + wl = \frac{vw^2}{(w+v)^2}q^2 + \frac{v^2w}{(w+v)^2}q^2 = \frac{vw}{(w+v)}q^2$

c. Show how you would use the principle of profit maximization to derive the profit function of this firm. You should show all of the work that would allow you to arrive at the final form: $\Pi(P,v,w) = \frac{P^2(v+w)}{4vw}$.

$\Pi = P(k^{.5} + l^{.5}) - vk - wl$ First order conditions for a maximum are

$$\frac{\partial \Pi}{\partial k} = .5Pk^{-.5} - v = 0$$

$$\frac{\partial \Pi}{\partial l} = .5Pl^{-.5} - w = 0$$

$$So\ k = \frac{P^2}{4v^2} \quad l = \frac{P^2}{4w^2}$$

$$\Pi = \frac{P^2}{2v} + \frac{P^2}{2w} - \frac{P^2}{4v} - \frac{P^2}{4w} = \frac{P^2(v+w)}{4vw}$$

d. Use the cost function from part b to calculate the supply function for this firm.

$$MC = \frac{\partial C}{\partial q} = \frac{2vw}{(v+w)} q = P\ or\ q = \frac{P(v+w)}{2vw}.$$

e. Use the profit function from part c to calculate the supply function for this firm. Show that the result here agrees with the result obtained in part d.

By the envelope theorem: $q = \frac{\partial \Pi}{\partial P} = \frac{P(v+w)}{2vw}$. So the two approaches agree.

f. Use the supply functions derived in parts d and e to show that $\frac{\partial q}{\partial P} > 0$ and $\frac{\partial q}{\partial w} < 0$. Explain what these conclusions mean about this firm's supply curve.

The first result is obvious from the supply function. This means that the firm supplies more when price rises. To get the second derivative:
$\frac{\partial q}{\partial w} = \frac{2vwP - 2vP(w+v)}{4v^2w^2} = \frac{-P}{2w^2} < 0$. The supply function shifts leftward for an increase in *w*.

g. The production function for this firm looks sort of like an infinite elasticity of substitution production function of the form $q = k + l$ except that it has exponents on the inputs. How would the firm's response to a change in the wage on labor demand differ from what it would be if the function were of the infinite elasticity of substitution type?

The linear production function exhibits constant returns to scale. Hence the firm may use only labor or only capital depending on costs. In the case in this problem, diminishing returns prevents that and induces less than perfect input substitutability.

h. Without making any calculations, explain how you would disaggregate this firm's labor demand response to a change in the wage into substitution and output effect components.

The disaggregation can be made by first calculating the total effect of a wage change on labor demand as the negative of the second derivative or the profit function. Now the contingent wage effect on labor demand can be calculated as the second derivative of the cost function.

2. We have seen that the profit maximization hypotheses will lead to a profit function of a price-taking firm of the form $\Pi^* = \Pi(P, v, w)$. This question will ask you to explain several of the characteristics of this function.

a. Why is this function homogeneous of degree 1 in P, v, and w ?

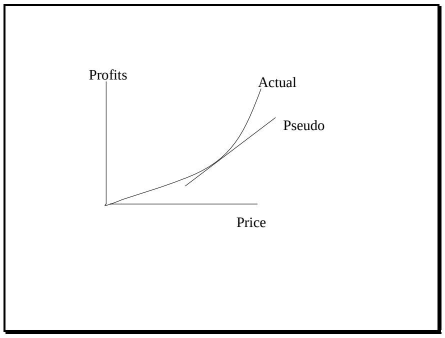
A doubling of these three prices will double profits providing k and l do not change. But these inputs will not change because the firm has no incentive to change output level nor to change the input mix (because relative prices have not changed).

b. A little known theorem in Chapter 2 states that if a function is homogeneous of degree k, its partial derivatives are homogeneous of degree k-1. What are the implications of this mathematical property for the firm's supply function and for its input demand functions? For each function describe the economic relevance of your result.

Because the supply function is the derivative of the profit function, the function is homogeneous of degree zero in all prices. That is, a doubling of all output and input prices will not change the firm's supply behavior. A "pure" inflation does not affect output or input demand.

By Shephard's Lemma the firm's input demand functions are also homogeneous of degree zero in all prices. A doubling of all prices will not affect the amounts of inputs demanded – the marginal value product of labor and the wage shift upward by equal amounts.

c. Profit functions are convex in output price. Provide a graphical/mathematical argument of why this is so. Then describe the implications of this mathematical fact for the shape firm's supply function.



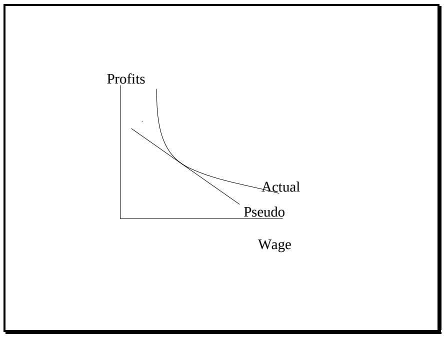
This graph shows a pseudo profit function in which nothing varies but price. The actual profit function must lie above this tangency by the assumption of profit maximization – the firm can always do no worse than the pseudo profits. Because this curve is convex, $\frac{\partial^2 \Pi}{\partial P^2} = \frac{\partial q}{\partial P} > 0$. That is, the supply curve is positively sloped.

d. Describe precisely how the results of part c are relevant to the theory of commodity price-stabilization.

The average of profits from a fluctuating price will be greater than profits at a stabilized average price because any chord joining two points on the profit function will lie above the function.

e. Provide a graphical/mathematical argument of why the profit function is concave in input prices. State the relevance of this result for the shape of the firm's input demand functions.

The pseudo profit function below shows profits if the firs changes nothing when wages change. Actual profits lie everywhere above this line. Hence the actual profit function is convex in *w*. That is $\frac{\partial^2 \Pi}{\partial w^2} = -\frac{\partial l}{\partial w} > 0$ *Hence* $\frac{\partial l}{\partial w} < 0$. That is, any input demand curve is downward sloping.



3. The following questions refer to the paper "Measuring Input Substitution in Thrifts…" by K.J. Stiroh

a. This paper is about input substitution in a particular industry. Explain why the author is interested in this topic – what are some of the larger implications of input substitutability (or lack thereof) for any industry?

The author is interested in devising the proper way to measure how firms in the thrift industry are able to substitute among inputs. Input substitution in the face of changing relative input prices is an important way of mitigating the effect on total cost of increases in the price of particular inputs. Firms that can adapt in this way will be more viable in an industry than those who cannot.

b. The author points out that there is some ambiguity in how to define the elasticity of substitution between inputs when there are more than two inputs. Without going into the math, provide an explanation of what the nature of the problem is (your reading from the text may help on this).

When one looks at changes in the relative use of two inputs in response to changing relative prices one must decide what to assume about the third input. One approach would be to hold the third input constant. Another approach would be to hold the price of the third input constant, but allow its quantity of usage to vary in response to changes in the relative prices of the other two. This is roughly the difference between the Allen/Uzawa and Morishima definitions.

c. The author claims that the Allen/Uzawa definition in Equation (2) is symmetric and leads directly to Equation (3). Show the logic underlying both of these statements.

All of this follows from Young's theorem (that $C_{ij} = C_{ji}$) and from Shephard's Lemma. The symmetry of definition (2) is clear – all of the i's and j's can easily be reversed.

Equation 3 follows because $\sigma_{jk} = \dfrac{C\dfrac{\partial x_j}{\partial p_k}}{x_j x_k} \quad (Shephard's\ Lemma) = \dfrac{\dfrac{\partial x_j}{\partial p_k} \cdot \dfrac{p_k}{x_j}}{\dfrac{p_k x_k}{C}} = \dfrac{\varepsilon_{jk}}{s_k}$.

d. Some authors claim that the Morishima elasticity is "biased" in that it is overly likely to identify inputs as substitutes. How does Equation (6) hint at this possibility?

The Equation is $M_{jk} = \varepsilon_{kj} - \varepsilon_{jj}$. In words, the Morishima measure is the difference between the cross-price elasticity of input demand and the own price elasticity. But $\varepsilon_{jj} < 0$ because of the convexity of the profit function. Hence M will always have a positive component – a "bias" toward showing substitutability relative to the A/U definition which has the same sign as ε_{kj} .

e. Stiroh's primary conclusions are reported in Table 5. What does this table show?

The table shows that the A/U and M elasticities are generally similar. But the main finding is that both measures of elasticity are larger for the low cost firms than for the high cost firms especially with respect to substitutions for IBLs (interest bearing liabilities (i.e. accounts). Perhaps that is the reason for these thrifts low costs.