Solutions to the Algebra problems on the Comprehensive Examination of February 2, 2007

1. Let G be a group, and let $H \subseteq G$ be a subgroup. Let L be the set of left cosets of H in G, and let R be the set of right cosets of H in G. That is,

$$L = \{aH : a \in G\}, \qquad \text{and} \qquad R = \{Ha : a \in G\}.$$

Define the function $f : L \to R$ by $f(aH) = Ha^{-1}$.

(a) Prove that f is well-defined.

Solution: Given $a, b \in G$ such that $aH = bH$, we show that $f(aH) = f(bH)$, or equivalently $Ha^{-1} = Hb^{-1}$. First, note that $aH = bH \Rightarrow a^{-1}b \in H$. But then $a^{-1}(b^{-1})^{-1} = a^{-1}b \in H$, so $Ha^{-1} = Hb^{-1}$ as desired. QED

(b) Prove that f is onto.

Solution: Given $Ha \in R$, $f((a^{-1})H) = H(a^{-1})^{-1} = Ha$ so f is onto.✓

(*Note: you may* **not** *assume that L or R is finite, and you may* **not** *assume that H is normal in G.*)

2. Fix an integer $n \geq 2$, and write S_n for the permutation group on n letters. Let

$$\phi : S_n \to G$$

be a homomorphism, where G is a group of odd order. (I.e., G is a finite group with an odd number of elements.)

(a) Prove that every **transposition** (i.e., 2-cycle) $\tau \in S_n$ is in $\ker \phi$.
That is, prove that $\phi(\tau) = e$.

Solution: Given a transposition τ, trivially $o(\tau) = 2$. Thus since ϕ is a homomorphism, $e_G = \phi(e_{Sn}) = \phi(\tau^2) = \phi(\tau)\phi(\tau)$, so either $\phi(\tau) = e_G$ or $o(\phi(\tau)) = 2$. But if $o(\phi(\tau)) = 2$, then $2 \big| |G|$, which is a contradiction because $|G|$ is odd. Thus, $\phi(\tau) = e_G$.✓

(b) Prove that ϕ is the trivial homomorphism; i.e., prove that for all $\sigma \in S_n$, we have $\phi(\sigma) = e$.

Solution: We know that S_n is generated by transpositions. Given $\sigma \in S_n$, write σ as a product of transpositions $\sigma = \tau_1\tau_2 \cdots \tau_m$. Then by part (a), $\phi(\sigma) = \phi(\tau_1)\phi(\tau_2) \cdots \phi(\tau_m) = (e_G)^m = e_G$ as desired. QED

3. Let R be a ring with unity, and let $I \subseteq R$ be a subset.

(a) Define what it means for I to be an **ideal** of R.

Solution: $I \subseteq R$ is an ideal of R if $(I, +)$ is a subgroup of $(R, +)$ and $\forall x \in I$, $r \in R$, $xr \in I$ and $rx \in I$.

(b) Recall that a **unit** is an element $u \in R$ that has a multiplicative inverse $v \in R$. If I is an ideal and contains a unit, prove that $I = R$.

Solution: Let $u \in I$ be a unit in I. Since $I \subseteq R$, it suffices to show that $R \subseteq I$. So, given $r \in R$, by property of ideals $ru \in I$. But then also by property of ideals, $r = (ru)u^{-1} \in I$ as desired. QED

4. Let $R = \mathbb{Z}[x]$ be the ring of polynomials (in one variable) with integer coefficients. Note that the constant polynomial 2 and the degree one polynomial x are both elements of R. Define

$$I = \{2f + xg : f, g \in R\}.$$

(a) Prove that I is an ideal of R.

Solution: First, we show that $(I, +)$ is a subgroup of $(R, +)$:
(I nonempty) Let $h(x) = 2 + x$. Then $h \in I$ (with $f(x) = g(x) = 1$).✓
(I closed under $+$) Given $h_1, h_2 \in I$, $h_1 = 2f_1 + xg_1$ and $h_2 = 2f_2 + xg_2$ for some $f_1, f_2, g_1, g_2 \in R$. Then $h_1 + h_2 = 2(f_1 + f_2) + x(g_1 + g_2)$ so $h_1 + h_2 \in I$.✓
(I closed under negatives) Given $h \in I$, $h = 2f + xg$ for some $f, g \in R$. Then $-h = 2(-f) + x(-g)$ so $-h \in I$.✓
Thus $(I, +)$ is a subgroup of $(R, +)$ ✓
Now given $\zeta \in R$, $h \in I$, $h = 2f + xg$ for some $f, g \in R$. Then $\zeta h = h\zeta = 2(f\zeta) + x(g\zeta)$ so $\zeta h, h\zeta \in I$.✓
Thus, I is an ideal of R. QED

(b) Prove that

$$I = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_i \in \mathbb{Z} \text{ and } a_0 \text{ is even}\}.$$

That is, prove that I consists of exactly those polynomials in R with even constant term.

Solution: ($\subseteq$): Given $h \in I$, $h = 2f + xg$ for some $f, g \in R$. Letting $f(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^m$ and $g(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^k$, we have $h(x) = 2b_0 + (2b_1 + c_0)x + H(x)$, where $H(x)$ is some polynomial in which each term has degree at least 2. Thus h has an even constant term.✓
($\supseteq$): Given $h(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in R$ such that a_0 is even, let $f(x) = a_0/2$ (note $f(x) \in R$ because a_0 is even so $a_0/2 \in \mathbb{Z}$) and $g(x) = a_1 + a_2x + a_3x^2 + \cdots + a_nx^{n-1}$ (note $g \in R$). Then $h = 2f + xg$, $f, g \in R$ so $h \in I$ as desired.✓
Thus $I = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_i \in \mathbb{Z} \text{ and } a_0 \text{ is even}\}$ as desired. QED