Solutions to the Algebra problems on the Comprehensive Examination of January 28, 2011

1. **(30 points).** Let G be a group, and let $H \subseteq G$ be a subset of G.

   (a) (15 points). Suppose that $H \subseteq G$ is a subgroup with the property that for every $x, y \in G$, we have $xyx^{-1}y^{-1} \in H$. Prove that H is a **normal** subgroup of G.

   Solution: Given $g \in G$, $h \in H$, by the given property of H $ghg^{-1}h^{-1} \in H$. Since H is closed under $*$, $ghg^{-1} = (ghg^{-1}h^{-1})h \in H$ as desired.✓

   (b) (15 points). Given G and H as in part (c), prove that the quotient group G/H is abelian.

   Solution: Given $Ha, Hb \in G/H$, we need to show $HaHb = HbHa$. This is equivalent to $Hab = Hba$, which by the criterion for equality of cosets is equivalent to $(ab)(ba)^{-1} \in H$. However, $(ab)(ba)^{-1} = aba^{-1}b^{-1}$, and $aba^{-1}b^{-1} \in H$ by the given property of H. Thus $Hab = Hba$ so G/H is abelian, as desired.✓

2. **(20 points).** Let G be a finite group, and suppose that there is an element $a \in G$ with the property that $a^{-1} = a$ but a is not the identity. Prove that G has an even number of elements.

   Solution: It is given that $a^2 = aa^{-1} = e$, so $o(a) \leq 2$. Since $a \neq e$, $o(a) \neq 1$ so $o(a) = 2$. Since $o(a) \mid |G|$ by Lagrange's Theorem, we have $2 \mid |G|$, so $|G|$ is even as desired.✓

3. **(25 points).** Consider the group S_6 of permutations of the set $\{1, 2, 3, 4, 5, 6\}$. Let $\sigma \in S_6$ be the permutation

$$\sigma = (3\ 6\ 5)(1\ 6\ 3)(1\ 4\ 5\ 2)(4\ 6).$$

   (a) (8 points). Write σ as a product of **disjoint** cycles.

   Solution: $\sigma = (1\ 4\ 6\ 3)(2\ 5)$

   (b) (8 points). Compute the **order** of σ.

   Solution: The order of σ is the lcm of the orders of each individual cycle (and the order of an n-cycle is n): $o(\sigma) = \text{lcm}(4, 2) = 4$.

   (c) (9 points). Is σ **even** or **odd**? Why?

   Solution: $\sigma = (1\ 4\ 6\ 3)(2\ 5) = (1\ 4)(4\ 6)(6\ 3)(2\ 5)$, so σ is the product of 4 transpositions so it is even.

4. **(25 points).** Let R be a ring.

(a) (10 points). Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.

If you use other terms like "closed" or "coset" or "subgroup" or "subring" or "maximal" in your definition, you must define those terms as well.

Solution: $I \subseteq R$ is an ideal of R if $(I, +)$ is a subgroup of $(R, +)$ and $\forall x \in I,\ r \in R,\ xr \in I$ and $rx \in I$. If G is a group, a subgroup is a set $H \subseteq G$ such that H forms a group under G's group operation.

(b) (15 points). Let R be the ring of **differentiable** functions $f : \mathbb{R} \to \mathbb{R}$ from the real line to itself, under the usual operations of multiplication and addition of functions. Let

$$I = \{f \in R : f(3) = f'(3) = 0\}.$$

Prove that I is an ideal of R.

Solution: First, we show that $(I, +)$ is a subgroup of $(R, +)$:

(I nonempty) Let e be the zero function, which is defined by $e(x) = 0$ for all $x \in \mathbb{R}$. Then $e(3) = 0$ and $e'(3) = 0$, so $e \in I$, so $I \neq \emptyset$.✓

(I closed under $+$) Given $f, g \in I$, $(f + g)(3) = f(3) + g(3) = 0 + 0 = 0$ and $(f + g)'(3) = f'(3) + g'(3) = 0 + 0 = 0$ so $f + g \in I$.✓

(I closed under negatives) Given $f \in I$, $(-f)(3) = -f(3) = 0$ and $(-f)'(3) = -f'(3) = 0$ so $-f \in I$.✓

Thus $(I, +)$ is a subgroup of $(R, +)$.✓

Now given $r \in R$, $f \in I$, $(rf)(3) = r(3)f(3) = r(3)0 = 0$ and $(rf)'(3) = r'(3)f(3) + r(3)f'(3) = r'(3)0 + r(3)0 = 0$, so $rf \in I$. Similarly, $(fr)(3) = f(3)r(3) = 0r(3) = 0$ and $(fr)'(3) = f'(3)r(3) + f(3)r'(3) = 0r(3) + 0r'(3) = 0$, so $fr \in I$.✓

Thus, I is an ideal of R. QED