Problem set 3

1 Horizon

$$D_{horizon} = ct \qquad D_{galaxy} = D_{now}\, R(t)$$

$$f \equiv D_{horizon} / D_{galaxy} = \frac{c\, t}{D_{now}\, R(t)}$$

$$\frac{df}{dt} = \dot{f} = \frac{c}{D_{now}} \frac{d}{dt}\left[\frac{t}{R(t)}\right]$$

$$\dot{f} = \frac{c}{D_{now}}\left[\frac{1}{R(t)} - R^{-2}(t)\, t\, \dot{R}(t)\right]$$

for this to be positive we need

$$\frac{1}{R} > \frac{t\dot{R}}{R^2} \quad \text{so} \quad 1 > t\,\dot{R}/R$$

but $\dot{R}/R$ is $H = \frac{1}{t_H}$ so this

needs $1 > \frac{t}{T_H}$ ← true age of universe and we know this is so

T_H ← Hubble age

(2) do galaxies accelerate? (2)

a) $k=0$: $R(t) = \alpha t^{2/3}$ where $\alpha = \left[\frac{3A}{2}\right]^{2/3}$

and the distance to any galaxy

is $r(t) = r(\text{now})\, R(t)$

so that galaxy's velocity is

$$v = \dot{r} = r_{now}\, \dot{R}(t) = \frac{2}{3} r_{now}\, \alpha\, t^{-\frac{1}{3}}$$

as time gets bigger, v gets <u>less</u>

so the galaxy is <u>decelerating</u>.

another way (either way is fine):

find $\dot{v} = \ddot{r} = \text{accel}$

we get $\text{accel} = \left(-\frac{1}{3}\right)\left(\frac{2}{3}\right) r_{now}\, \alpha\, t^{-\frac{4}{3}}$

this is negative, meaning <u>deceleration</u>

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(6) for any k at all, just remember the first friedman eq

$$\ddot{R} = \underbrace{-}\ \tfrac{4}{3}\pi G \rho(t_o)\ \frac{1}{R^2}$$

the minus sign means deceleration!

(3) Connection between Ω and k

divide the 2nd eq by R^2 to get

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2 = \frac{8\pi G}{3}\frac{\rho(t)}{R^2} - \frac{k}{R^2}$$

evaluating this now we set $R(t_o) = 1$

$$H^2(t_o) = \frac{8\pi G}{3}\rho(t_o) - k$$

write $\rho(t_o) = \Omega \rho_{crit}$ where

$$\rho_{crit} = \frac{3H^2(now)}{8\pi G}$$

we get

$$k = H^2(t_o)\left[\Omega - 1\right]$$

Synthesis of Everything

Late times

$$R(t) = \left[\frac{3}{2} A t\right]^{2/3}$$

$$A^{*} = \left[\frac{8\pi G \rho(t_0)}{3}\right]^{\frac{1}{2}} \quad \text{but for } k=0,$$

$$\rho(t_0) = \rho_{critical} = \frac{3H^2}{8\pi G} \quad \text{so}$$

$$A = \left[\frac{8\pi G}{3}\,\frac{3H^2}{8\pi G}\right]^{1/2} = H = \frac{1}{t_H} \leftarrow \text{Hubble age}$$

Then $$R(t) = \left[\frac{3}{2}\,\frac{t}{t_H}\right]^{2/3} = \left[\frac{t \text{ in seconds}}{3.15\times10^{17}}\right]^{2/3}$$

$$\cancel{\rho(t) = \frac{\rho(t_0)}{R^3(t)}}$$

$$R(t) = 2.17\times10^{-12}\, t^{2/3}_{\text{in seconds}}$$

Then
$$T(t) = \frac{2.7}{R(t)} = \frac{1.24\times10^{12}}{t^{2/3}_{\text{in seconds}}} \;{}^{\circ}K$$

and, since $\rho R^3 = \text{const} = \rho_{crit}$

and $$\rho_{crit} = \frac{3H^2}{8\pi G} = \frac{3}{8\pi G t_H^2} = 8.0\times10^{-30} \text{ g/cc}$$

so $\rho(t) = \frac{8.0 \times 10^{-32}}{R^3(t)} \frac{g}{cc} = 7.83 \times 10^5 \frac{1}{t^2_{\text{in sec}}} \frac{g}{cc}$

<u>Early times</u> $[t < 100{,}000$ YRS $= 3.15 \times 10^{12}$ sec$]$

$$R(t) = \left[\frac{16 \pi G \rho(t_0)}{3}\right]^{\frac{1}{4}} t^{\frac{1}{2}} = 1.73 \times 10^{-9} \sqrt{t_{\text{in sec}}}$$

$[$and $\rho(t_0) = \rho_{crit}]$

so

$$T(t) = \frac{2.7}{R} = 1.56 \times 10^{9} \Big/ \sqrt{t_{\text{in sec}}} \quad {}^\circ K$$

$$\rho(t) = \frac{8.0 \times 10^{-30}}{R^3} = 1.55 \times 10^{-3} \left[\frac{1}{t_{\text{in sec}}}\right]^{3/2} \frac{g}{cc}$$

in what follows, you may get slightly different answers based on how you [or I] round numbers off. Don't worry about it.

milestones

now we find this by setting $R_{now} = 1$; we find

$$t^{2/3} = \frac{R(t)}{2.17 \times 10^{-12}} = \frac{1}{2.17 \times 10^{-12}} = 4.61 \times 10^{11}$$

$t = 9.93$ billion years $= 3.13 \times 10^{17}$ sec

$\rho = \rho_{crit} = 8 \times 10^{-30}$ $\qquad T = 2.7\ K$

Formation of Earth

4.5 billion years ago, so $t = (9.93 - 4.5)$ billion

~~[illegible]~~

$t = 5.43$ billion yrs $= 1.71 \times 10^{17}$ sec

$R = 0.65$

$T = 4.15\ ^\circ K$ $\qquad \rho = 2.68 \times 10^{-29}\ g/cc$

Formation of Milky Way

9 billion years ago, i.e. $t = 0.93$ billion years
$= 2.93 \times 10^{16}$ sec

$R = 0.2$ $\qquad T = 13.5\ ^\circ K$ $\qquad \rho = 9.12 \times 10^{-28}\ \frac{g}{cc}$

time = 1 year $= 3.15 \times 10^{7}$ sec

$$T = \frac{1.56 \times 10^{9}}{\sqrt{3.15 \times 10^{7}}} = 278{,}000\ ^\circ K$$

$\rho = 8.77 \times 10^{-15}$ g/cc

density = water [1 g/cc]

since @ t = 1 year the density is << water, we know this is for still earlier times and we use the "early times" solution

$$\rho = 1.55 \times 10^{-3} / t^{3/2} = 1 \quad \text{sc}$$

time $= 1.35 \times 10^{-2}$ sec

$$T = 1.56 \times 10^{9} / \sqrt{\text{time}} = 13.5 \text{ billion } ^\circ K$$

neutron density

mass of a neutron is 1.66×10^{-24} g

radius $\approx$ 10^{-13} cm so

$$\text{density} = \frac{\text{mass}}{\text{volume}} = 3.96 \times 10^{14} \text{ g/cc}$$

obviously we need to use the "early times" solution: the density has this value when times given by

$$\rho = 3.96 \times 10^{14} = \frac{1.55 \times 10^{-3}}{t^{3/2}}$$

$t = 2.55 \times 10^{-12}$ seconds (!)

$T = 7.77 \times 10^{14}$ °K (!)

temperature = 212 °F = 373 °K

it's not obvious whether this falls in the "early" or "late" period: you have to plug in and see. It turns out that it is "late" sec $T = 373 = \frac{1.24 \times 10^{12}}{t_{sec}^{2/3}}$ °K

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$t = 1.92 \times 10^{14}$ sec = 6.08 million years

$\rho = 2.12 \times 10^{-23}$ g/cc