Suppose you are modeling battery lifetimes with an exponential distribution. So, \( X_1, \ldots, X_n \) is a random sample with \( f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \) \( (X_i \sim \text{Exp}(\beta)) \).

a. Using all \( n \) obs., provide an unbiased pt. estimate for \( \beta \). (Show it is unbiased)

b. Derive the MSE of your pt. estimate from a.

c. What is the distribution of \( \sum_{i=1}^{n} X_i \)?

d. Looking @ c., derive a pivotal quantity for use in estimating \( \beta \) with a CI. What is the distribution of your pivotal?

e. Using either your answer to d., or a version of it, you should be able to get a \( \chi^2 \) dist. for the pivotal. Use this pivotal and obtain a 95% CI for \( \beta \) when \( \bar{X} = 23.45 \) hrs for a RS of \( n=10 \) batteries.
Math Stat Handout Solutions

\[ X_1, \ldots, X_n \sim \text{Exp}(\beta) \]

a. Try \( \bar{X} \). \( E(\bar{X}) = \frac{1}{n} E(\sum X_i) = \frac{1}{n} \sum E(X_i) = \frac{1}{n} \cdot n\beta = \beta \)

\( \bar{X} \) is unbiased for \( \beta \).

b. MSE(\( \bar{X} \)) = \( \text{Var}(\bar{X}) + (\text{Bias}(\bar{X}))^2 \) = \( \text{Var}(\bar{X}) = \frac{\beta^2}{n} \)

c. By rules on adding Gammas (b/c Exp(\( \beta \)) = Gamma(1, \beta))

\[ Y = \sum X_i \sim \text{Gamma}(n, \beta) \]

d. If we eliminated dependence on \( \beta \) we could get \( \text{Gamma}(n, \frac{\beta}{\beta}) = (1 - \beta t)^{-n} = M_Y(t) \)

directing by \( \beta \) would do... \( U = \frac{Y}{\beta} \)

\[ M_U(t) = (1 - \beta (\frac{t}{\beta}))^{-n} \Rightarrow (1 - t)^{-n} \sim \text{Gamma}(n, 1) \]

So \( \frac{Y}{\beta} \) is a pivot \( \frac{\sum X_i}{\beta} \)

e. To get a \( \chi^2 \) dist, that's Gamma\( \left( \frac{n}{2}, 2 \right) \) so, we need to multiply by 2.

\[ V = \frac{2 \sum X_i}{\beta} \]

\[ M_V(t) = (1 - \beta (\frac{2}{\beta} t))^{-n} = (1 - 2t)^{-n} \]

\[ \sum X_i \sim \text{Gamma}(n, \frac{\beta}{\beta}) \Rightarrow \sum X_i \sim \text{Gamma}(n, 2) \]

\[ \sum X_i \sim \text{Gamma}(n, 2) \Rightarrow \chi^2(2n) \]

\[ P(a \leq V \leq b) = .95 \Rightarrow P\left( \frac{\chi^2(20)}{975} \leq \frac{2 \sum X_i}{\beta} \leq \frac{\chi^2(20)}{.005} \right) = .95 \]

\[ P\left( 9.59083 \leq \frac{469}{\beta} \leq 34.1696 \right) \Rightarrow P\left( \frac{1}{34.1696} \leq \beta \leq \frac{1}{9.59083} \right) \]

\[ \Rightarrow 95\% \text{ CI is } \left( \frac{469}{34.1696}, \frac{469}{9.59083} \right) = (13.72565, 48.90088) \]