Math 211, Multivariable Calculus, Fall 2011
Midterm II Practice Exam 2 Solutions

1. *In a certain ski slope, the elevation (in meters) of the land at the points with coordinates* (x, y) *(also measured in meters) is given by the function*

$$h(x,y) = 1000 - \frac{x^2}{100} - \frac{y^2}{50}$$

Which direction could a skier at the point $(25\sqrt{3}, -12.5)$ *go in order to descend at an angle of* 45 *degree? Give your answer in the form of the angle between the direction the skier should head and due North. (Assume that the vector* $\langle 1, 0\rangle$ *points due North and* $\langle 0, 1\rangle$ *points due East.)*

To descend at an angle of 45 degrees we want the slope in the direction the skier goes to be -1. That is, the directional derivative of h in that direction should be -1.

The gradient vector of h is

$$\nabla h = \langle -x/50, -y/25\rangle.$$

At the given point, we have

$$\nabla h(25\sqrt{3}, -12.5) = \langle -\sqrt{3}/2, 1/2\rangle.$$

We therefore want to find a unit vector $\mathbf{u}$ such that

$$\langle -\sqrt{3}/2, 1/2\rangle \cdot \mathbf{u} = -1.$$

If the angle between $\mathbf{u}$ and the gradient vector is θ, we would have

$$|\langle -\sqrt{3}/2, 1/2\rangle||\mathbf{u}| \cos\theta = -1.$$

Now $|\mathbf{u}| = 1$ since $\mathbf{u}$ must be a unit vector, and $|\langle -\sqrt{3}/2, 1/2\rangle| = 1$ by calculation, so we must have

$$\cos\theta = -1$$

so $\theta = \pi$. This tells us that $\mathbf{u}$ should point exactly opposite the gradient vector, that is, in the direction of

$$\langle \sqrt{3}/2, -1/2\rangle.$$

We can now draw a diagram to find the angle between this vector and due North (i.e. the vector $\langle 1, 0\rangle$. We see this is 120 degrees. So the skier should head at an angle of 120 degrees from due North.

2. *Find the linear approximation to the function*

$$f(x,y) = \cos(3xy) - \sin(3xy)$$

at the point $(1, \pi)$ *and use it to estimate the value of* $f(1, \pi - 0.01)$.

We have

$$\frac{\partial f}{\partial x} = -3y\sin(3xy) - 3y\cos(3xy)$$

and

$$\frac{\partial f}{\partial y} = -3x\sin(3xy) - 3x\cos(3xy).$$

Therefore

$$\frac{\partial f}{\partial x}(1,\pi) = -3\pi\sin(3\pi) - 3\pi\cos(3\pi) = 3\pi$$

and

$$\frac{\partial f}{\partial y}(1,\pi) = -3\sin(3\pi) - 3\cos(3\pi) = 3.$$

Since we also have $f(1,\pi) = -1$, the linear approximation is

$$l(x,y) = -1 + 3\pi(x-1) + 3(y-\pi).$$

Therefore, we get the estimate

$$f(1,\pi - 0.01) \approx -1 + 3(-0.01) = -1.03.$$

3. *In each of the following cases, find a function f of two variables that satisfies the given condition. (You should justify your answers as precisely as possible.)*

   (a) *f is defined at all points and is continuous at all points except* $(0,0)$

   (b) *f is continuous at all points and is differentiable at all points except* $(0,0)$

   (a) There are many possibilities. For example, the function

$$f(x,y) = \begin{cases} 1 & \text{if } (x,y) \neq (0,0); \\ 2 & \text{if } (x,y) = (0,0). \end{cases}$$

   This is continuous at points not equal to $(0,0)$ because in a region around any such point, it is the constant, continuous function 1. But the limit of f as $(x,y) \to (0,0)$ is 1 which is not equal to $f(0,0)$, so f is not continuous at $(0,0)$. (Notice that the explanation must have reference to the definition of being continuous, namely that the limit as (x,y) approaches $(0,0)$ either is or is not equal to $f(0,0)$.)

   (b) Again there are many possibilities but it is harder to justify. A good one to keep in mind is

$$f(x,y) = (x^2 + y^2)^{1/3}.$$

   This is defined everywhere by a single formula involving power functions and addition, so is continuous everywhere. At points other than $(0,0)$, the partial derivatives are given by

$$\frac{\partial f}{\partial x} = \frac{2x}{3(x^2+y^2)^{2/3}}$$

   and

$$\frac{\partial f}{\partial y} = \frac{2y}{3(x^2+y^2)^{2/3}}.$$

   These are both continuous everywhere they are defined, which is everywhere except $(0,0)$. So, the partial derivatives of f are continuous at and around any point that is not $(0,0)$, so f is differentiable there.

On the other hand, if we try to find the partial derivatives at $(0,0)$, we get

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h^{2/3}}{h} = \lim_{h\to 0} \frac{1}{h^{1/3}}.$$

This limit does not exist because $h^{1/3} \to 0$ as $h \to 0$. Therefore, the partial derivative at $(0,0)$ does not exist and so f is not differentiable at $(0,0)$.

4. *A particle moves with position vector at time t given by*

$$\mathbf{r}(t) = \langle \sin(2t), 2t, \cos(2t+\pi)\rangle.$$

Find the tangential and normal components of the acceleration of the particle.

The tangential component of acceleration is given by

$$a_T = v'$$

where $v = |\mathbf{v}|$ is the speed of the path. We have

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2\cos(2t), 2, -2\sin(2t+\pi)\rangle$$

and so

$$v(t) = \sqrt{4\cos^2(2t) + 4 + 4\sin^2(2t+\pi)}.$$

Now $\sin(2t+\pi) = -\sin(2t)$, so $\sin^2(2t+\pi) = \sin^2(2t)$. Therefore, the speed is

$$v(t) = \sqrt{4\cos^2(2t) + 4 + 4\sin^2(2t)} = \sqrt{8} = 2\sqrt{2}.$$

This is constant, so the tangential component of the acceleration is zero.

The normal component of acceleration is given by

$$a_N = \kappa v^2$$

where κ is the curvature. Here it is easiest to use the formula

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

We have

$$\mathbf{r}''(t) = \langle -4\sin(2t), 0, -4\cos(2t+\pi)\rangle$$

and so

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle -8\cos(2t+\pi), -8\sin^2(2t) - 8\cos^2(2t), 8\sin(2t)\rangle$$

and so

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{64\cos^2(2t) + 64 + 64\sin^2(2t)} = \sqrt{128} = 8\sqrt{2}.$$

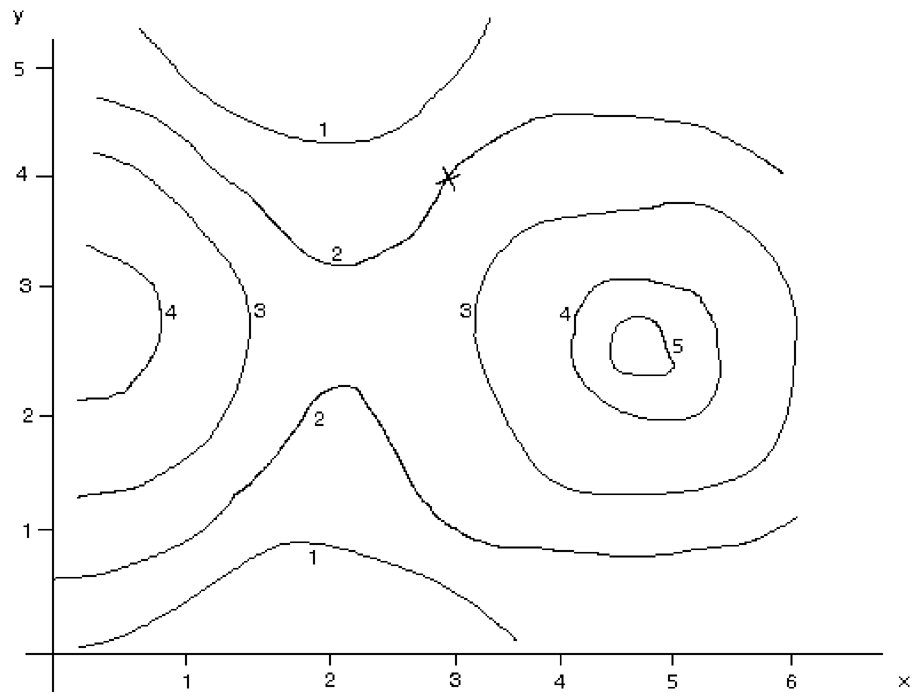
The curvature is therefore

$$\kappa(t) = \frac{8\sqrt{2}}{(2\sqrt{2})^3} = \frac{1}{2}.$$

The normal component of acceleration is therefore

$$a_N = \kappa v^2 = 4.$$

5. *The following diagram shows a selection of level curves for a function $f(x, y)$:*



(a) *Estimate the value of each of the partial derivatives of f at the point $(3, 4)$ (marked with a cross).*

(b) *Give a vector* $\mathbf{u}$ *such that the directional derivative at $(3, 4)$ of f in the direction* $\mathbf{u}$ *is approximately equal to* 0*.*

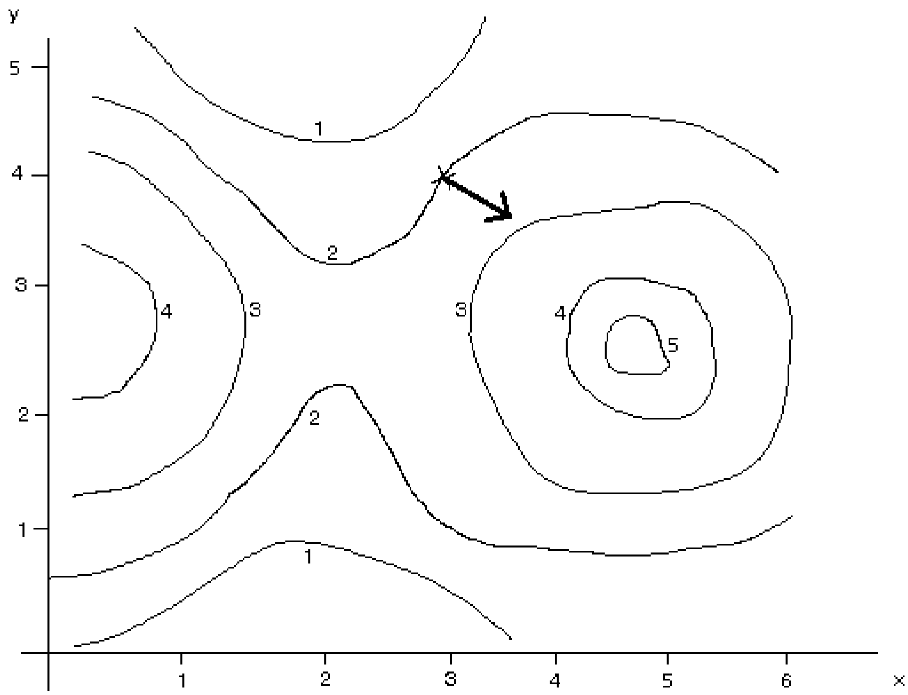
(c) *Draw on the above diagram an arrow to represent the gradient vector of f at the point $(3, 4)$. (Only the direction of your arrow is important.)*

(d) *Draw a graph to illustrate the $y = 4$ cross-section of the function f. (Be as accurate as possible based on the information given.) What is the connection between this graph and one of your answers to part (a)?*

(a) $\dfrac{\partial f}{\partial x}(3, 4) \approx 0.5$, $\dfrac{\partial f}{\partial y}(3, 4) \approx -1$. (In (c) below we realize that these are not that accurate, as the ratio is not quite right, but this is good enough based on the diagram.)

(b) The vector $\mathbf{u}$ should be tangent to the level curve at $(3, 4)$, for example $\mathbf{u} \approx \langle 1, 2 \rangle$.

(c) The gradient vector at $(3, 4)$ should be perpendicular to the level curve at $(3, 4)$, and in the direction of increasing f. (This actually suggests that our guesses in part (a) were a bit off as the absolute value of $\frac{\partial f}{\partial x}$ should be greater than that of $\frac{\partial f}{\partial y}$.)



(d) The connection is that the slope of the tangent line to the cross-section at $x = 3$ should be equal to the partial derivative $\frac{\partial f}{\partial x}(3, 4)$ which is about 0.5.