Math 28 Spring 2008: Exam 2

Instructions: Each problem is scored out of 10 points for a total of 40 points. You may not use any outside materials(eg. notes or books). You have 50 minutes to complete this exam.

Problem 1.

(a) Let $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$. State the definition for f to be uniformly continuous on A.

(b) Which of the following functions are uniformly continuous on $[0, \infty)$?

(i) $f(x) = \sin(x^2)$

(ii) $f(x) = \frac{1}{x+1}$

Proof.

(a) A function $f : A \to \mathbb{R}$ is uniformly continuous on A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $x, y \in A$, $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$.

(b) (i) No. Let the sequence of points (x_n) be defined by $x_n = 2n\pi$ and the sequence of points (y_n) be defined by $y_n = 2n\pi$. Then we have

$$|f(x_n) - f(y_n)| = \left| f(\sqrt{2n\pi}) - f(\sqrt{2n\pi + \frac{\pi}{2}}) \right| = |\sin(2n\pi) - \sin(2n\pi + \pi/2)| = 1.$$

Also notice that $|x_n - y_n| \to 0$ as $n \to \infty$. So for any $\epsilon \leq 1$ and for any $\delta > 0$ there exists an $N \in \mathbb{N}$ such that $|x_n - y_n| < \delta$, but we have that $|f(x_N) - f(y_N)| = 1 \geq \epsilon$.

(ii) Yes. Let $\epsilon < 0$ and let $\delta = \epsilon$ Then for $x, y \in A$ with $|x - y| < \delta$ we have

$$\begin{aligned} \left| \frac{1}{1+x} - \frac{1}{y+1} \right| &= \left| \frac{y - x}{(1+x)(1+y)} \right| \\ &\leq \frac{|x-y|}{1} = \delta = \epsilon. \end{aligned}$$

where the inequality comes from the fact that $x, y \in [0, \infty)$ and so $\frac{1}{1+x}, \frac{1}{1+y} \geq 1$.

□

Problem 2. Let C be the Cantor set on $[0, 1]$ obtained in the standard way by successively removing the middle third of each interval. Define $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C. \end{cases}$$

(a) Show that g is discontinuous at every point in C.

(b) Show that g is continuous at every point not in C.

Proof. We will use the topological criterion for continuity.

(a) Let $c \in C$ and let $\epsilon = \frac{1}{2}$. Then for every $\delta > 0$, the neighborhood $V_\delta(c)$ in not a subset of C (since we proved in class that C contains no intervals). Thus there exists a point $x \in V_\delta(c)$ with $x \notin C$ and hence $g(x) = 0 \notin V_\epsilon(g(c))$.

(b) Let $c \notin C$ and let $\epsilon > 0$. Since C we proved in class that is closed, its complement is open. Hence, there exists a $\delta > 0$ such that $V_\delta(c) \subseteq C^c$. Take any $x \in V_\delta(c)$, then $x \in C^c$ and hence $g(x) = 0$. Hence we have $x \in V_\delta(c)$ implies $g(x) \in V_\epsilon(g(c))$ and g is continuous at c.

□

Problem 3.

(a) State the Generalized Mean Value Theorem.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and suppose that f' is bounded. Show that f is uniformly continuous.

Proof.

(a) If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ where

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

If $g'(x)$ is never 0 on (a, b), then the conclusion can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(b) We are given that $f'(x)$ is bounded. So let $M > 0$ be a bound for $f'(x)$. Let $\epsilon > 0$ and let $\delta = \frac{\epsilon}{M}$. Consider $x > y \in \mathbb{R}$ such that $|x - y| < \delta$. Then we have by the Mean Value Theorem that there exists a $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$

and hence

$$\frac{|f(y) - f(x)|}{|y - x|} = M$$

so we have

$$|f(y) - f(x)| = M\,|y - x| < M\frac{\epsilon}{M} = \epsilon.$$

□

Problem 4.

(a) State the definition for a function $f : A \to \mathbb{R}$ to be differentiable on an interval A.

(b) Let $f : [-1, 1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show that f is differentiable, but that its derivative is unbounded.

Proof.

(a) Let $f : A \to \mathbb{R}$ be a function defined on an interval A. Given $c \in A$, the *derivative* of f at c is defined by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided the limit exists. If $f'(c)$ exists for all be points c in A, then we say that f is *differentiable on* A

(b) To show differentiable, we first note that x^2 and $\sin x$ are continuous differentiable functions and hence their product is differentiable. Since $\sin \frac{1}{x^2}$ is defined everywhere but $x = 0$, we can use the standard differentiation rules for $x \neq 0$ and only need to consider the limit definition for the case $x = 0$. Consider

$$\lim_{x \to 0} \frac{f(x) - 0}{x} = \lim_{x \to 0} x \sin \frac{1}{x^2}.$$

This is bounded by

$$-x \leq x \sin \frac{1}{x^2} \leq x$$

so by the Squeeze Theorem the limit approaches 0. Hence $f'(0)$ also exists.

The derivative is given by

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

where the case $x \neq 0$ is from standard differentiation rules (product rule and chain rule).

To see that the derivative is unbounded, consider the sequence $x_n = \frac{1}{\sqrt{(2n+1)\pi}}$ with $(x_n) \to 0$ and

$$\lim_{n \to \infty} -\frac{1}{x} \cos \frac{1}{x^2} = \lim_{n \to \ infty} -\frac{1}{\sqrt{(2n+1)}}(-1) = \lim_{n \to \infty} \frac{1}{\sqrt{(2n+1)}} = \infty.$$

Hence we have values of $f'(x)$ which can be arbitrarily large as $x \to 0$.

□