Math 12 Fall 2009: Exam 1

Name:

Instructions: You may not use any outside materials (eg. notes or calculators). Each problem is scored out of 8 points for a total of 32 points. You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Problem 1. Evaluate the following limits

(a)
$$\lim_{x\to 0} \frac{e^{2x} - x - 1}{x(x+1)}$$

(b)
$$\lim_{x\to 0} (\cot x)\ln(1+2x)$$

Proof.

(a) This is $\frac{0}{0}$ so we apply l"Höpital's Rule to get

$$\lim_{x\to 0} \frac{e^{2x} - x - 1}{x(x+1)} = \lim_{x\to\infty} \frac{2e^{2x} - 1}{2x+1} = \frac{2-1}{1} = 1.$$

(b) We have $\infty \cdot 0$ so we make it a fraction and apply l'Höpital's Rule to

$$\lim_{x\to 0} (\cot x)\ln(1+2x) = \lim_{x\to 0} \frac{\ln(1+2x)}{\tan x} = \lim_{x\to 0} \frac{2}{(1+2x)(\sec^2 x)}$$
$$= \lim_{x\to 0} \frac{2\cos^2 x}{1+2x} = 2.$$

□

Problem 2. Evaluate the following definite integrals.

(a)

$$\int_0^{\pi/2} \frac{\csc x \cot x}{1+\cot^2 x}\, dx$$

(b)

$$\int_0^{\ln 2} 8\cosh^2 x\, dx$$

Proof.

(a) We simplify with trigonometric identities then integrate.

$$\int_0^{\pi/2} \frac{\csc x \cot x}{1+\cot^2 x} dx = \int_0^{\pi/2} \frac{\cot x}{\csc x} dx = \int_0^{\pi/2} \cos x dx = \sin x]_0^{\pi/2} = 1.$$

(b) We apply the definition of $\cosh x$.

$$\begin{aligned}\int_0^{\ln 2} 8\cosh x^2 dx &= \int_0^{\ln 2} 8\left(\frac{e^x+e^{-x}}{2}\right)^2 dx \\ &= 2\int_0^{\ln 2} e^{2x} + 2 + e^{-2x} dx \\ &= e^{2x} + 4x - e^{-2x}]_0^{\ln 2} \\ &= 4 + 4\ln 2 - \frac{1}{4} - (1+0-1) \\ &= 3\frac{3}{4} + 4\ln 2.\end{aligned}$$

□

Problem 3. Evaluate the following indefinite integrals.

(a)
$$\int \frac{\ln x}{x^{3/2}} dx$$

(b)
$$\int \frac{x^3}{x^2+2x+1} dx$$

Proof. (a) We apply integration by parts with $u = \ln x, dv = x^{-3/2}$ and so $du = x^{-1}, v = -2x^{-1/2}$.

$$\begin{aligned} \int \frac{\ln x}{x^{3/2}} dx &= \ln x(-2x^{-1/2}) - \int -2x^{-1/2}x^{-1} dx \\ &= -2x^{-1/2}\ln x + \int 2x^{-3/2} dx \\ &= -2x^{-1/2}\ln x - 4x^{-1/2} + C \end{aligned}$$

(b) We first apply long division and the partial fractions.

$$\begin{aligned} \int \frac{x^3}{x^2+2x+1} dx &= \int x - 2 + \frac{3x+2}{x^2+2x+1} dx \\ &= \frac{x^2}{2} - 2x + \int \frac{3}{x+1} - \frac{1}{(x+1)^2} dx \\ &= \frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1} + C \end{aligned}$$

□

Problem 4. Find the area under the curve $y = (x-2)\sqrt{x^2-4x+8}$ from $[4, 2+2\sqrt{3}]$.

Proof. We have

$$A = \int_4^{2+2\sqrt{3}} (x-2)\sqrt{8+2x-x^2}dx = \int_4^{2+2\sqrt{3}} (x-2)\sqrt{(x-2)^2+4}dx$$

and use $x-2 = 2\tan\theta, dx = 2\sec^2\theta d\theta$. We adjust the bounds as $2 = 2\tan\theta$ and $2\sqrt{3} = 2\tan\theta$ to get

$$\begin{aligned} A &= \int_{\pi/4}^{\pi/3} 2\tan\theta\sqrt{4(\tan^2\theta+1)}(2\sec^2\theta d\theta \\ &= \int_{\pi/4}^{\pi/3} 2\tan\theta(4\sec^3\theta)d\theta \\ &= 8\int_{\pi/4}^{\pi/3} \tan\theta\sec^3\theta d\theta. \end{aligned}$$

Setting $u = \sec\theta, du = \sec\theta\tan\theta$ we have

$$\begin{aligned} A &= 8\int_{\theta=\pi/4}^{\theta=\pi/3} u^2 du = 8\left(\frac{u^3}{3}\right)_{\theta=\pi/4}^{\theta=\pi/3} \\ &= 8\left(\frac{\sec^3\theta}{3}\right)_{\pi/4}^{\pi/3} \\ &= 8\left(\frac{8}{3} - \frac{2\sqrt{2}}{3}\right) \\ &= \frac{64-16\sqrt{2}}{3}. \end{aligned}$$

□