Math 12 Fall 2009: Exam 2

October 26, 2009

Name:

Instructions: There are 4 questions on this exam each of which is scored out of 8 points for a total of 32 points. You may not use any outside materials (eg. notes or books). You may use your calculator **ONLY** for problem 1. You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Problem 1.

(a) Write but do not evaluate the integral for the arclength of $y = \frac{x^2}{2} + 1$ for $1 \leq x \leq 3$.

(b) How many terms are needed to approximate the integral to within $\frac{1}{1000}$. You may use any method.

Proof.

(a) We have $\frac{dy}{dx} = x$ and so

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + x^2.$$

So we have

$$s = \int_1^3 \sqrt{1+x^2}dx.$$

(b) Since Simpson's Rule requires a fourth derivative this solution is only for Midpoint and Trapezoidal Rules.

To use Midpoint Rule we need the second derivative of $\sqrt{1+x^2}$ which is computed as

$$\begin{aligned} f' &= \frac{x}{\sqrt{1+x^2}} \\ f'' &= \frac{\sqrt{1+x^2} - \frac{x \cdot 2x}{2\sqrt{1+x^2}}}{1+x^2} \\ &= \frac{1}{\sqrt{1+x^2}} - \frac{x^2}{(1+x^2)^{3/2}} \\ &= \frac{1+x^2-x^2}{(1+x^2)^{3/2}} = \frac{1}{(1+x^2)^{3/2}}. \end{aligned}$$

an upper bound for $x \in [1,3]$ is given by

$$f'' \leq 1.$$

So we know that

$$|EM_n| \leq \frac{K(b-a)^3}{24n^2} = \frac{(3-1)^3}{24n^2} = \frac{8}{24n^2}$$

We need to have

$$\frac{1}{1000} > \frac{1}{3n^2}$$

so we get

$$n \geq 19 > 18.26.$$

Using Trapezoidal Rule we still have $K = 1$ and we use the estimate

$$|ET_n| \leq \frac{K(b-a)^3}{12n^2} = \frac{(3-1)^3}{12n^2} = \frac{8}{12n^2}$$

We need to have

$$\frac{1}{1000} > \frac{2}{3n^2}$$

so we get

$$n \geq 26 > 25.82.$$

□

Problem 2. For the following two improper integrals, determine whether they converge or diverge. If the integral converges, find its value.

(a)

$$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

(b)

$$\int_2^\infty \frac{1}{\sqrt[3]{x-1}} dx$$

Proof.

(a) We evaluate with the substitution $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}$

$$\begin{aligned}\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx &= \lim_{a\to 0^+} \int_a^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx \\ &= \lim_{a\to 0^+} (2e^{\sqrt{x}})_a^1 \\ &= \lim_{a\to 0^+} (2e - 2e^{\sqrt{a}}) \\ &= 2e - 2.\end{aligned}$$

(b) We can integrate this with $u = x - 1, du = dx$ to get

$$\begin{aligned}\int_2^\infty \frac{1}{\sqrt[3]{x-1}} dx &= \lim_{b\to\infty} \left(\frac{3(x-1)^{2/3}}{2}\right)_2^b \\ &= \lim_{b\to\infty} \frac{3b^{2/3}}{2} - \frac{3}{2} = \infty.\end{aligned}$$

We could also use comparison. Since $x - 1 \geq 1$, making the denominator larger will decrease the function so we have

$$\frac{1}{\sqrt[3]{x-1}} \geq \frac{1}{\sqrt[3]{x}} = x^{-1/3}.$$

So we have

$$\int_2^\infty \frac{1}{\sqrt[3]{x-1}} dx \geq \int_2^\infty x^{-1/3} dx.$$

By the p-test, $\int_2^\infty x^{-1/3}$ diverges since $\frac{1}{3} < 1$. So we have that the integral diverges by comparison. Or we can explicitly compute

$$\begin{aligned}\int_2^\infty x^{-1/3} dx &= \lim_{b\to\infty} \left(\frac{3}{2} x^{2/3}\right)_2^b \\ &= \lim_{b\to\infty} \left(\frac{3b^{3/2}}{2} - \frac{3 \cdot 2^{3/2}}{2}\right) = \infty.\end{aligned}$$

□

Problem 3. Consider the region between $y = \sqrt{x}$ and $y = x^3$ rotated around $x = 2$.

(a) Write but do not evaluate an integral for the volume using disk or washer method.

(b) Write but do not evaluate an integral for the volume using cylindrical shell method.

(c) Evaluate one of the two integrals to compute the volume.

Proof. The intersection points are $(0,0)$ and $(1,1)$.

(a) We have washers and the integral is

$$V = \pi \int_0^1 (2-y^2)^2 - (2-\sqrt[3]{y})^2 dy.$$

(b) In cylindrical shells we have

$$V = 2\pi \int_0^1 (2-x)(\sqrt{x} - x^3) dx.$$

(c) Evaluating the washers we have

$$\begin{aligned} V &= \pi \int_0^1 (2-y^2)^2 - (2-\sqrt[3]{y})^2 dy \\ &= \pi \int_0^1 (4-4y^2+y^4) - (4-4y^{1/3}+y^{2/3}) dy \\ &= \pi \int_0^1 -4y^2 + y^4 + 4y^{1/3} - y^{2/3} dy \\ &= \pi \left(-\frac{4}{3}y^3 + \frac{1}{5}y^5 + 3y^{4/3} - \frac{3}{5}y^{5/3} \right)\Bigg|_0^1 \\ &= \pi \left(-\frac{4}{3} + \frac{1}{5} + 3 - \frac{3}{5} \right) \\ &= \frac{38\pi}{30} = \frac{19\pi}{15}. \end{aligned}$$

Evaluating the Cylindrical Shells we have

$$\begin{aligned} V &= 2\pi \int_0^1 (2-x)(\sqrt{x} - x^3) dx \\ &= 2\pi \int_0^1 2x^{1/2} - 2x^3 - x^{3/2} + x^4 dx \\ &= 2\pi \left(\frac{4}{3}x^{3/2} - \frac{1}{2}x^4 - \frac{2}{5}x^{5/2} + \frac{1}{5}x^5 \right)\Bigg|_0^1 \\ &= 2\pi \left(\frac{4}{3} - \frac{1}{2} - \frac{2}{5} + \frac{1}{5} \right) \\ &= \frac{38\pi}{30} = \frac{19\pi}{15}. \end{aligned}$$

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Problem 4. Consider the surface $y = \ln x$ for $0 < x \leq 1$ rotated around the y-axis.

(a) Write but do not evaluate an integral to determine the surface area.

(b) Show that the surface area integral converges.

(c) (Bonus) Find the surface area.

Proof.

(a) Integrating with respect to dx we have $y = \ln x$ and so $1 + (y')^2 = 1 + \frac{1}{x^2} = \frac{x^2+1}{x^2}$

$$SA = \lim_{a \to 0+} \int_a^1 2\pi x \sqrt{1 + \frac{1}{x^2}} dx = \lim_{a \to 0+} \int_a^1 2\pi \sqrt{x^2 + 1} dx$$

Integrating with respect to dy we have $x = e^y$ and so $1 + (x')^2 = 1 + e^{2y}$.

$$SA = \int_{-\infty}^0 2\pi e^y \sqrt{1 + e^{2y}} dy$$

(b) We can either evaluate the integral (see below) or perform a comparison. Comparing the dx integral, since $x^2 + 1 > 1$, we have

$$\sqrt{x^2 + 1} \leq x^2 + 1$$

and so

$$\lim_{a \to 0+} \int_a^1 2\pi x \sqrt{\frac{x^2 + 1}{x^2}} dx \leq \lim_{a \to 0+} \int_a^1 2\pi(x^2 + 1) dx$$

We can evaluate the larger integral as

$$\lim_{a \to 0+} \int_a^1 2\pi(x^2 + 1) dx = 2\pi \lim_{a \to 0+} \left(\frac{x^3}{3} + x \right)_a^1 = \frac{8\pi}{3}.$$

Since the larger integral converges, so does the surface area.

We could also compare the dy integral as

$$2\pi e^y \sqrt{1 + e^{2y}} \leq 2\pi e^y (1 + e^{2y})$$

since $1 + e^{2y} \geq 1$ to get

$$\int_{-\infty}^0 2\pi e^y \sqrt{1 + e^{2y}} dy \leq \int_{-\infty}^0 2\pi e^y (1 + e^{2y}) dy.$$

We compute the larger integral as

$$\begin{aligned} \int_{-\infty}^0 2\pi e^y + e^{3y} dy &= \lim_{a \to -\infty} \int_a^0 2\pi e^y + e^{3y} dy \\ &= \lim_{a \to -\infty} \left(e^y + \frac{e^{3y}}{3} \right)_a^0 \\ &= \lim_{a \to -\infty} 1 + \frac{1}{3} - \left(e^a + \frac{e^{3a}}{3} \right) = \frac{4}{3}. \end{aligned}$$

Since the larger integral converges so does the smaller integral.

(c) Integrating with respect to dx we have

$$SA = 2\pi \lim_{a\to 0+} \int_a^1 \sqrt{x^2+1}dx.$$

Making the trig substitution $x = \tan\theta, dx = \sec^2\theta$, we have

$$SA = 2\pi \lim_{a\to 0+} \int_a^{\pi/4} \sqrt{1+\tan^2\theta}\sec^2\theta d\theta = 2\pi \lim_{a\to 0+} \int_a^{\pi/4} \sec^3\theta d\theta.$$

Using integration by parts with $dv = \sec^2\theta, u = \sec\theta$ we get

$$\begin{aligned} SA &= 2\pi \lim_{a\to 0+} \left(\sec\theta\tan\theta]_a^{\pi/4} - \int_a^{\pi/4} \sec\theta\tan^2\theta d\theta \right) \\ &= 2\pi \lim_{a\to 0+} \left(\sec\theta\tan\theta]_a^{\pi/4} + \int_a^{\pi/4} \sec\theta d\theta - \int_a^{\pi/4} \sec^3\theta d\theta \right). \end{aligned}$$

We use the identity $1+\tan^2\theta = \sec^2\theta$ and notice that we now have $2\pi\int\sec^3\theta d\theta$ on both sides, so we add it to both sides and divide by 2 to get

$$\begin{aligned} SA &= 2\pi \lim_{a\to 0+} \frac{1}{2} \left(\sec\theta\tan\theta]_a^{\pi/4} + \ln|\sec\theta+\tan\theta|]_a^{\pi/4} \right) \\ &= \pi\left((\sqrt{2}-0) + \ln(\sqrt{2}+1) - \ln 1 \right) \\ &= \pi\left(\sqrt{2} + \ln(\sqrt{2}+1) \right) \end{aligned}$$

Integrating with respect to dy we have $x = e^y$ and so $1+(x')^2 = 1+e^{2y}$.

$$\begin{aligned} SA &= \lim_{a\to -\infty} \int_a^0 2\pi e^y \sqrt{1+e^{2y}}dy \\ &= 2\pi \lim_{a\to -\infty} \int_a^0 e^y \sqrt{1+e^{2y}}dy. \end{aligned}$$

With the substitution $u = e^y$ we have

$$SA = 2\pi \lim_{a\to 0^+} \int_a^1 \sqrt{1+u^2}du.$$

and now we proceed as above.

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