Solutions to the Calculus and Linear Algebra problems on the Comprehensive Examination of January 29, 2010

Solutions to Problems 1-5 and 10 are omitted since they involve topics no longer covered on the Comprehensive Examination.

6. [15 points] Evaluate the following integrals:

   (a) $\int_0^{\ln 10} \int_{e^x}^{10} \frac{1}{\ln y}\, dy\, dx.$

   Solution: We need to switch the order of integration. The region of integration over lies between $y = e^x$ and $y = 10$ for $0 \leq x \leq \ln 10$. This is the same as the region between $x = 0$ and $x = \ln y$ for $1 \leq y \leq 10$. Thus

$$\begin{aligned}\int_0^{\ln 10} \int_{e^x}^{10} \frac{1}{\ln y}\, dy\, dx &= \int_1^{10} \int_0^{\ln y} \frac{1}{\ln y}\, dx\, dy \\ &= \int_1^{10} \frac{x}{\ln y}\Big|_{x=0}^{\ln y} dy = \int_1^{10} \frac{\ln y}{\ln y}\, dy \\ &= \int_1^{10} dy = 9\end{aligned}$$

   (b) $\int_C \cos(x^2)dx + (3xy^2 + x^3)dy$, where C is the circle $x^2 + y^2 = 4$, oriented counter-clockwise.

   Solution: Just apply Greens's Theorem, letting D be the domain enclosed by C:

$$\int_C \cos(x^2)dx + (3xy^2 + x^3)dy = \iint_D (3y^2 + 3x^2) - 0\, dA$$

   Now integrate on D using polar coordinates:

$$\int_0^{2\pi} \int_0^2 3r^2 \cdot r\, dr\, d\theta = 3(2\pi)\Big(\frac{r^4}{4}\Big|_{r=0}^2\Big) = 24\pi$$

7. [10 points] Find the volume of the region that is inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{3x^2 + 3y^2}$.

   Solution: We'll be working with spherical coordinates here. The inside-the-sphere constraint is pretty obvious: it just means that $\rho \leq 2$. The cone constraint is a bound on ϕ. To find the specific bound, it can be helpful to just consider the xz-plane. Above the cone on that plane means that $z \geq \sqrt{3}x$, which converted to spherical means $\tan(\phi) \leq 1/\sqrt{3}$, or $\phi \leq \arctan(1/\sqrt{3}) = \pi/6$. Any θ satisfies the constraints, so integrating:

$$\begin{aligned}
V &= \iiint\limits_V dV \\
&= \int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta \\
&= 2\pi \Big(\frac{\rho^3}{3}\Big|_0^2 \Big) \Big(-\cos(\phi)\Big|_0^{\pi/6} \Big) \\
&= \frac{16\pi}{3} \Big(1 - \cos(\pi/6) \Big) \\
&= \frac{16\pi}{3} \Big(1 - \frac{\sqrt{3}}{2} \Big)
\end{aligned}$$

To remember the cosine and tangent of $\pi/6$, take an equilateral triangle of side 2 and divide it down the middle to get a right triangle with a 30° angle at top, adjacent $= \sqrt{3}$, opposite $= 1$, and hypotenuse $= 2$.

8. [12 points] Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^3 + 4xy^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Find $f_x(0,0)$ and $f_y(0,0)$.

   Solution: Use the definition of partial derivative:

$$f_x(0,0) = \lim_{h\to 0} \frac{\frac{h^3+4(h)(0^2)}{h^2+0^2}}{h} = \lim_{h\to 0} \frac{h^3}{h^3} = 1$$

$$f_y(0,0) = \lim_{h\to 0} \frac{\frac{0^3+4(0)(h^2)}{0^2+h^2}}{h} = \lim_{h\to 0} \frac{0}{h^3} = 0$$

   (b) Is f differentiable at (0,0)? Justify your answer.

   Solution: Will not need to prove differentiability in the new comps.

9. [10 points] Find the point on the plane $2x - y + 2z = 16$ that is nearest the origin.

   Solution: We use Lagrange multipliers here to find the min of $f(x,y,z) = x^2 + y^2 + z^2$ (distance from the origin) with the constraint $g(x,y) = 2x - y + 2z = 16$. To find the points to test, we solve the system $\nabla f(x,y) = \lambda \nabla g(x,y)$ (so $2x = \lambda(2)$, $2y = \lambda(-1)$, and $2z = \lambda(2)$) and $g(x,y) = 16$. To solve this system, note that the first three constraints turn into $x = \lambda$, $y = -\lambda/2$, and $z = \lambda$. Substituting into the last equation gives $2\lambda+\lambda/2+2\lambda = 16 \Rightarrow \lambda = 32/9$, which in turn gives the point $(32/9, -16/9, 32/9)$ (for a distance of 256/9). Note that this is the only critical point, so it must be the global minimum and not a max, since the distance from the origin at (for example) $(10^{10}, -16, -10^{10})$ is clearly much greater than 256/9.

11. [8 points] Let $\mathbf{A}$ be a square matrix and let α be a scalar that is NOT an eigenvalue of $\mathbf{A}$. Suppose that μ is an eigenvalue for the matrix $\mathbf{B} = (\mathbf{A} - \alpha\mathbf{I})^{-1}$ with corresponding eigenvector $\mathbf{v}$. Prove that $\mathbf{v}$ is also an eigenvector for $\mathbf{A}$ and find a formula for the corresponding eigenvalue of $\mathbf{A}$ in terms of μ and α.

    Solution: We start with the given eigenvalue equation for $(\mathbf{A} - \alpha\mathbf{I})^{-\mathbf{1}}$ and the solution proceeds from there:

    $$(\mathbf{A} - \alpha\mathbf{I})^{-\mathbf{1}}\mathbf{v} = \mu\mathbf{v} \;\Rightarrow\; (\mathbf{A} - \alpha\mathbf{I})(\mathbf{A} - \alpha\mathbf{I})^{-\mathbf{1}}\mathbf{v} = (\mathbf{A} - \alpha\mathbf{I})\mu\mathbf{v} \;\Rightarrow\; \mathbf{v} = \mu\mathbf{A}\mathbf{v} - \mu\alpha\mathbf{I}\mathbf{v}$$

    The last equation implies in particular that $\mathbf{v} = \mu(\mathbf{A}\mathbf{v} - \alpha\mathbf{I}\mathbf{v})$, so that $\mu \neq 0$ since $\mathbf{v} \neq \mathbf{0}$ (eigenvectors are nonzero). Hence we can continue the implications

    $$\mathbf{v} = \mu\mathbf{A}\mathbf{v} - \mu\alpha\mathbf{I}\mathbf{v} \;\Rightarrow\; \mu\mathbf{A}\mathbf{v} = \mathbf{v} + \mu\alpha\mathbf{I}\mathbf{v} \;\Rightarrow$$

    $$\mu\mathbf{A}\mathbf{v} = (\mathbf{1} + \mu\alpha)\mathbf{v} \;\Rightarrow\; \mathbf{A}\mathbf{v} = (\mathbf{1}/\mu + \alpha)\mathbf{v}$$

    Thus $\mathbf{v}$ is an eigenvector for $\mathbf{A}$, with eigenvalue $1/\mu + \alpha$, as desired. QED

12. [10 points] Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V. Suppose T is one-to-one (injective). Prove that if $\{v_1, ..., v_n\}$ is a basis for V, then $\{T(v_1), ..., T(v_n)\}$ is also a basis for V.

    Solution: First, we show that $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent.
    Given scalars $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) = 0$:
    Since T is linear, $T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) = 0$. Note that T is linear, so $T(0)$ also equals 0. Since T is one-to-one, this means that $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$. We know that $\{v_1, \ldots, v_n\}$ is a basis, so it is linearly independent, which means $\alpha_i = 0$ $\forall i \in \{1, \ldots, n\}$. But that's what we were trying to prove! So $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent , as desired.

    Now note that if $\{v_1, \ldots, v_n\}$ is a basis for V, then V has dimension n. Thus since $\{T(v_1), \ldots, T(v_n)\}$ is a set of n linearly independent vectors, it is a basis for V (in an n-dimensional vector space, n vectors span $\Leftrightarrow$ they are linearly independent $\Leftrightarrow$ they form a basis). QED

13. [15 points] Let $T : P_2 \to \mathbb{R}^3$, where $P_2 = \{a + bt + ct^2 : a, b, c \in \mathbb{R}\}$, be defined by

    $$T(p) = \begin{bmatrix} p(1) \\ p(1) \\ p'(1) \end{bmatrix}.$$

Here $p'(t)$ is the derivative of the polynomial $p(t)$. Determine the null space (kernel) and range of T.

Solution: It helps to represent T using the basis $\{1, t, t^2\}$:

$$T(p) = \begin{bmatrix} a+b+c \\ a+b+c \\ b+2c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Now row reduce:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

This gives the nullspace span $\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)$ in $\mathbb{R}^3$, which corresponds to span$(1 - 2t + t^2)$ in P_2. Thus $1 - 2t + t^2$ is a basis of the kernel of T.

Since T maps to $\mathbb{R}^3$, the range of T equals the column space of its matrix representation. The previous paragraph shows that the nullity is 1, so the rank-nullity theorem tells us that the rank of the matrix is $3 - 1 = 2$. Thus the column space has dimension 2. The first two columns are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Since these are clearly linearly independent (neither is a multiple of the other), they form a basis for the column space. Hence they are a basis of the range of T.