Introduction to Analysis
Constructing $\mathbb{R}$ from $\mathbb{Q}$

Definition. A subset $A \subset \mathbb{Q}$ is called a *cut* if it posses the following three properties

1. $A \neq \emptyset$ and $A \neq \mathbb{Q}$.
2. If $r \in A$, then also A contains every rational $q < r$.
3. A does not have a maximum, that is, if $r \in A$, then there exists an $s \in A$ with $r < s$.

Remark. Note that for a cut A, if $r \in A$ and $s \notin A$, then $r < s$. This is because all rational $q < r$ are in A. Hence, a rational number $s \notin A$ must satisfy $r < s$.

Example.

1. $C_r = \{t \in \mathbb{Q} \mid t < r\}$
2. $\{t \in \mathbb{Q} \mid t < \sqrt{2}\} = \{t \in \mathbb{Q} \mid t \leq \sqrt{2}$
3. Not a cut $\{t \in \mathbb{Q} \mid t \leq 2\}$.

Definition. The real numbers, denoted $\mathbb{R}$, is the set of all cuts in $\mathbb{Q}$.

1 Ordering on Cuts

Theorem. *(Ordering on $\mathbb{R}$) Inclusion of sets ($\subseteq$) is an ordering on cuts.*

Proof. We need to prove the three properties of an ordering:

1. For arbitrary cuts $A, B \in \mathbb{R}$ at least one of $A \subseteq B$ or $B \subseteq A$ is true.

   Proof. If $A \subseteq B$ then we are done, so assume not. We need to show that $B \subseteq A$. By assumption, there exists an element $a \in A$ such that $a \notin B$. Let $b \in B$ be arbitrary. Since $a \notin B$ the lemma implies that $b < a$ and by definition of a cut we must have $b \in A$. □

2. If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

   Proof. This is standard definition of equality of sets. □

3. (transitive) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

   Proof. Again, trivial since we are working with sets. □

□

2 Addition on Cuts

Theorem. *(Addition in $\mathbb{R}$) Given $A, B, C \in \mathbb{R}$ we define*

1. $A + B = \{a + b \mid a \in A, b \in B\}$ *such that* $A + B = B + A$ *and* $(A + B) + C = A + (B + C)$.
2. $O = \{t \in \mathbb{Q} \mid t < 0\}$ *such that* $A + O = A$.
3. $-A = \{r \in \mathbb{Q} \mid$ *there exists* $t \notin A$ *with* $t < -r\}$ *such that* $A + (-A) = O$
4. *If* $A \subseteq C$ *then* $A + B \subseteq A + C$

Proof. Let A, B be two cuts.

1. (Addition) We first need to see that $A + B$ is a cut, then we need to check that addition satisfies: commutativity, associativity, identities exist, and inverses exist.

   (a) ($A + B$ is a cut):

   i. First we see that $A + B$ is non-empty since A, B are non-empty. To see that $A + B \neq \mathbb{Q}$ we find an upper bound for $A + B$. Let $l_1 \notin A$ and $l_2 \notin B$. For any $a \in A$ and $b \in B$ we have that $a < l_1$ and $b < l_2$. Hence we have $l_1 + l_2$ as an upper bound of $A + B$.

   ii. Let $r = a + b \in A + B$. Let $s \in \mathbb{Q}$ such that $s < r$. Then we have $s < a + b$ and hence $s - b < a$. Hence $s - b \in A$. So we have $s = (s - b) + b \in A + B$.

   iii. To see that $A + B$ does not have a maximum, fix $c \in A + B$ and write $c = a + b$ for some $a \in A$ and $b \in B$. So there exists and $s \in A$ with $a < s$ and an $r \in B$ with $b < r$. So we have $c = a + b < s + r$.

   (b) (Commutative): We simply write

$$\begin{aligned} A + B &= \{a + b \mid a \in A, b \in B\} \\ &= \{b + a \mid a \in A, b \in B\} \\ &= B + A. \end{aligned}$$

   (c) (Associative): Let C be a third cut. Then we simply write

$$\begin{aligned} (A + B) + C &= \{(a + b) + c \mid a \in A, b \in B, c \in C\} \\ &= \{a + (b + c) \mid a \in A, b \in B, c \in C\} \\ &= A + (B + C). \end{aligned}$$

2. (Identity): Let $O = \{p \in \mathbb{Q} \mid p < 0\}$.

   (a) O is clearly a cut.

   (b) We need to show that $A + O \subseteq A$ and $A \subseteq A + O$.
   Let $a + b \in A + O$. Then we have $a + b < a \in A$. Similarly, let $a \in A$. Then for some $\epsilon > 0$, $s + \epsilon \in A$. Then $(s + \epsilon) - (-\epsilon) \in A + B$.

3. (Additive Inverse): Let $-A = \{r \in \mathbb{Q} \mid \exists t \notin A \text{ with } t < -r\}$.

   (a) ($-A$ is a cut): Note that for $A = \{r \in \mathbb{Q} \mid r < 1/2$ we have that $-A = \{r \in \mathbb{Q} \mid r < -1/2\}$.

i. Since $A \neq \mathbb{Q}$ there exists a $t \notin A$. Since $t < t+1$ we can conclude that $-(t+1) \in -A$ by the definition of $-A$, so we have $-A$ is non-empty.

ii. To show $-A \neq \mathbb{Q}$ we know that A is non-empty so there exists an $a \in A$. If $r \in -A$, there exists a $t \notin A$ with $t < -r$. Then $t \notin A$ implies that $a < t$ and hence $r < -a$. So $-A$ is bounded above by $-a$.

iii. Let $r \in -A$ and consider $s \in \mathbb{Q}$ with $s < r$. We need $s \in -A$. Because $r \in -A$ there exists a $t \notin A$ with $t < -r$. Since $s < r$ implies $-r < -s$ we have $t < -s$ and hence $s \in -A$.

iv. Let $r \in -A$, then there exists a $t \notin A$ with $t < -r$. By the density of $\mathbb{Q}$ we can choose an $s \in \mathbb{Q}$ such that $t < s < -r$. Hence $-s \in -A$ and because $r < -s$, we see that $-A$ does not possess a maximum.

(b) ($A + -A = O$): Let $r \in -A$. Because $-r \in -A$ we know there exists a $t \notin A$ with $t < -r$. Since $t \notin A$ we have $a < t$ for all $a \in A$, and hence $a < -r$. Thus, $a + r < 0$ and so $a + r \in O$ as needed to show that $A + (-A) \subseteq O$.

To show the reverse inclusion let $o \in O$ and we need to find an $a \in A$ and a $b \in -A$ such that $a + b = o$. Let $\epsilon = \frac{|o|}{2} = \frac{-o}{2}$. Now choose $t \in \mathbb{Q}$ such that $t \notin A$ and $t - \epsilon \in A$. $t \notin A$ implies that $-(t+\epsilon) \in -A$. So we have

$$-(t+\epsilon) + (t-\epsilon) = -2\epsilon = o.$$

4. (Respects ordering): Assume that $A \subseteq B$. Then consider $r = a + c \in A + C$. Then we have $r \in B + C$ since $a \in B$ by assumption.

□

3 Multiplication in $\mathbb{R}$

Theorem. *(Multiplication in $\mathbb{R}$) Let $A, B \geq O$ be in $\mathbb{R}$.*

1. *For $A, B \supseteq O$, $AB = \{ab \mid a \in A, b \in B \text{ with } a, b \geq 0\} \cup \{q \in \mathbb{Q}, q < 0\}$*
2. *$I = \{t \in \mathbb{Q} \mid t < 1\}$ such that $AI = A$.*
3. *For $A \neq O$, $A^{-1} = \{t \in \mathbb{Q} \mid \frac{1}{t} \notin A, \exists r \notin A s.t. r < \frac{1}{t}\} \cup \{t \in \mathbb{Q} \mid t \leq 0\}$ such that $AA^{-1} + I$.*
4. *$C(A + B) = CA + CB$.*
5. *$AC \supseteq O$.*

Remark. For negative multiplication and inverse, we take $-A$.

Proof.

1. (AB is a cut)

   (a) AB is not empty because all rational numbers $t < 0$ are in AB. Let s, t be upper bounds for A and B respectively. Then AB is bounded above by st, so AB cannot be all of $\mathbb{Q}$.

(b) Let $t \in AB$ be arbitrary and let $s \in \mathbb{Q}$ satisfy $s < t$. If $s < 0$ the $s \in AB$ by the definition of our product, so let $0 \le s < t$. So we have $t = ab$ with $a \in A$ and $b \in B$ and $a, b > 0$. Because $s < ab$ we have $\frac{s}{b} < a$ and hence $\frac{s}{b} \in A$. Then

$$s = \left(\frac{s}{b}\right)(b) \in AB.$$

(c) Let $t \in AB$. If $t < 0$ then $t < t/2$ and $t/2 \in AB$ since $t/2 < 0$ as well. If $t \geq 0$, then $t = ab$ for some $a \in A$ and $b \in B$. Applying the 3rd property of cuts to A, B yields $s \in A$ and $r \in B$ such that $a < s$ and $b < r$. So we have $sr \in AB$ and $ab < sr$.

2. (Multiplicative Identity):

(a) We first show that I is a cut

i. This is easy since $0 \in I$ and $2 \notin I$.

ii. Assume that $t \in I$, then for $s < t$ we have $s < 1$ and hence $s \in I$.

iii. Let $t \in I$. If $t < 0$, then $t < t/2 \in I$. if $t \geq 0$ then $t < \frac{t+1}{2} \in I$.

(b) ($AI \subseteq A$): Let $q \in AI$. Because $I \supseteq O$, then either $q < 0$ or $q = ab$ with $a, b \geq 0$ and $a \in A$ and $b < 1$. If $q < 0$, then $q \in A$ since $A \geq O$. In the other case, $q = ab < a$ and hence $ab \in A$.

(c) ($A \subseteq AI$): Let $a \in A$. If $a < 0$ then $a \in AI$ by the product of cuts. If $a \geq 0$, then we can pick a $t \in A$ with $a < t$. Hence $\frac{a}{t} < 1$ and hence $\frac{a}{t} \in I$. So we have

$$a = (t)\left(\frac{a}{t}\right) \in AI.$$

3. (Multiplicative Inverse)

(a) A^{-1} is a cut.

i. First, $-1 \in A^{-1}$ so it is not empty. So we need to show it is not all of $\mathbb{Q}$. Let $0 \le t \neq 0 \in A$ (this is possible, since $A \neq O$ and $A \supseteq O$). Then $\frac{1}{t} \notin A^{-1}$.

ii. Let $t \in A^{-1}$. Assume that $s < t$. Then we have $\frac{1}{t} < \frac{1}{s}$ and that $\frac{1}{t} \notin A$ so we must have $\frac{1}{s} \notin A$ and hence $\frac{1}{s} \in A^{-1}$.

iii. Let $t \in A^{-1}$. If $t < 0$, let $s = \frac{t}{2}$. So assume $t \geq 0$. Then $\frac{1}{t} \notin A$ and there exists and $r \notin A$ such that $r < \frac{1}{t}$. Since $\mathbb{Q}$ is dense, there exists a rational number $r < s < \frac{1}{t}$ that is also not in A (If $t = 0$ choose any $s > r$.). Then we have that $\frac{1}{s} \notin A$ and that $r < s$, so we have $\frac{1}{s} \in A^{-1}$ and $t < \frac{1}{s}$.

(b) $AA^{-1} \subseteq I$ Let $t \in AA^{-1}$ with $t \leq 0$, then $t \in I$. Let $a \in A$ and $b \in A^{-1}$ with $a, b > 0$. (Note that $a = b = 1$ is a contradiction, since that would imply that $\frac{1}{b} \in A$, so we must have one of less than 1.) Since $\frac{1}{b} \notin A$ we have $a < \frac{1}{b}$ and hence

$$ab < \left(\frac{1}{b}\right) b = 1$$

Hence $ab \in I$.

(c) $I \subseteq AA^{-1}$ If $A = I$ then $A^{-1} = I$ so there is nothing to show, so assume not.
WLOG assume that $A \subseteq I \subseteq A^{-1}$. Let $r \in I$. If $r \leq 0$, then $r \in AA^{-1}$. Now assume that we have $0 < r < 1$. There exits an $n \in \mathbb{N}$ such that $r^n \in A$ and $r^{n-1} \notin A$. If $\frac{1}{r^{n-1}} \in A^{-1}$ we have

$$r = r^n \left(\frac{1}{r^{n-1}} \right)$$

and are done. So assume that $\frac{1}{r^{n-1}} \notin A^{-1}$. This means that for every $\epsilon > 0$ we have $r^{n-1} - \epsilon \in A$. So we can find a rational $\epsilon > 0$ such that

$$r^n < r(r^{n-1} + \epsilon) < r^{n-1}$$

by the density of the rational numbers. So we have $\frac{1}{r^{n-1}+\epsilon} \in A^{-1}$ since $(r^{n-1} + \epsilon) \notin A$ and $r^{n-1} < r^{n-1} + \epsilon$. But also note that $r(r^{n-1} + \epsilon) \in A$.

$$r = r(r^{n-1} + \epsilon)\frac{1}{r^{n-1} + \epsilon} \in AA^{-1}$$

4. (Distributive Property): We show inclusion in both ways.
   (a) If $t \in C(A + B)$ then we can write $t = c(a + b)$. Since these are all rational numbers we have $t = ca + cb \in CA + CB$.
   (b) If $t \in CA + CB$ then we can write $t = ca + cb$. Since these are all rational number we have $t = c(a + b) \in C(A + B)$.
5. (Respects ordering): By the definition of AB this is trivial.

□

Remark. You will show for homework the embedding $\mathbb{Q} \subseteq \mathbb{R}$:
Let $C_r = \{t \in \mathbb{Q} \mid t < r\}$. Then we have

1. $C_r + C_s = C_{r+s}$
2. $C_r C_s = C_{rs}$.
3. etc.

For $r \in \mathbb{Q}$ this provides $\mathbb{Q}$ and a subfield of $\mathbb{R}$ (as cuts).

4 Least Upper Bounds

Definition. A set $\mathcal{A} \subseteq \mathbb{R}$ is *bounded above* if there exists a $B \in \mathbb{R}$ such that $A \leq B$ for all $A \in \mathcal{A}$. B is called an *upper bound* for A. $S \in R$ is a *least upper bound* for $\mathcal{A}$ if it satisfies

1. S is an upper bound for $\mathcal{A}$.
2. If B is any upper bound for $\mathcal{A}$, then $S \leq B$.

Theorem. *Let $S = \cup_{A \in \mathcal{A}} A$. S is the least upper bound for $\mathcal{A}$.*

Proof.

1. First we show that S is a cut.

   (a) Since $\mathcal{A} \neq \emptyset$, then $S \neq \emptyset$. Since $\mathcal{A}$ is bounded above by some cut B we have $S \leq B$. Since $B \neq \mathbb{Q}$ we have $S \neq \mathbb{Q}$.

   (b) Let $a \in S$ and $r \in \mathbb{Q}$ with $r < a$. Since $a \in S$, then $a \in A$ for some $A \in S$. Hence we have $r \in A$ and hence $r \in S$.

   (c) Let $a \in S$, then $a \in A$ for some $A \in S$. Therefore, there is an $r \in A$ such that $a < r$ and since $r \in A$, then $r \in S$.

2. Now we need to show that it is in fact a least upper bound.

   (a) S is an upper bound for $\mathcal{A}$ since $A \leq S$ for all $A \in \mathcal{A}$.

   (b) Let B be an arbitrary upper bound for $\mathcal{A}$. We need $S \leq B$. Let $s \in S$. There fore $s \in A$ for some $A \in \mathcal{A}$ and hence $s \in B$. Therefor $S \leq B$.

□

Since least upper bounds exists and $\mathbb{R}$ is complete. $\mathbb{R}$ is also an ordered field containing $\mathbb{Q}$.

Remark. Cantor used equivalence classes of Cauchy sequences to define $\mathbb{R}$. (two Cauchy sequences are equivalent if $(x_n - y_n) \to 0$.

Remark. Cuts may seem a weird concept compared to a decimal number, but with decimals how do you resolve the fact that $.5 = .49\overline{9}$.