Solutions to the Calculus and Linear Algebra problems on the Comprehensive Examination of January 28, 2011

Solutions to Problems 1–5 are omitted since they involve topics no longer covered on the Comprehensive Examination.

6. [8 points] Let C be the boundary (oriented counterclockwise) of the region under $y = e^x$ for $0 \le x \le 2$. Compute $\int_C 2xy^3\,dx + (xy + 3x^2y^2)\,dy$.

   Solution: Just apply Greens's Theorem. Let D be the region enclosed by C; then we have:

$$\begin{aligned}\int_C 2xy^3\,dx + (xy + 3x^2y^2)\,dy &= \iint_D \frac{\partial}{\partial x}(xy + 3x^2y^2) - \frac{\partial}{\partial y}(2xy^3)\,dA \\ &= \iint_D (y + 6xy^2) - 6xy^2\,dA = \iint_D y\,dA.\end{aligned}$$

   Now simply integrate y over D :

$$\begin{aligned}\iint_D y\,dA &= \int_0^2 \int_0^{e^x} y\,dy\,dx \\ &= \int_0^2 \frac{1}{2}y^2\Big|_0^{e^x} dx \\ &= \int_0^2 \frac{1}{2}e^{2x}\,dx \\ &= \frac{1}{2}\cdot\frac{1}{2}e^{2x}\Big|_0^2 = \frac{1}{4}(e^4 - 1).\end{aligned}$$

7. [15 points] Let V be the region in $\mathbb{R}^3$ inside the sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = 0$.

   (a) Express the volume of V in cartesian, cylindrical and spherical coordinates.

   Solution: We are working with a half sphere whose projection onto the xy-plane is $x^2 + y^2 \le 1$. The answer is straightforward for cartesian coordinates:

$$\text{Vol} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} 1\,dz\,dy\,dx.$$

   For cylindrical coordinates, the given constraint means $0 \le r \le 1$. Any θ satisfies the constraint. The top half of the sphere is $z = \sqrt{1 - r^2}$, so the integral becomes:

$$\text{Vol} = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta.$$

For spherical coordinates, the constraint for ρ is $0 \le \rho \le 1$. Any θ satisfies the constraint. Above the plane $z = 0$ means $0 \le \phi \le \pi/2$. Therefore the integral is:

$$\text{Vol} = \int_0^{2\pi}\int_0^1\int_0^{\pi/2} \rho^2 \sin\phi\,d\phi\,d\rho\,d\theta.$$

(b) Evaluate one of the integrals found in part (a).

Solution: We can choose any of these integrals to evaluate, but clearly it will be easier to use cylindrical coordinates or spherical coordinates because they involve less square root calculation. Below we give the solutions for both.

For cylindrical coordinates:

$$\begin{aligned}
\text{Vol} &= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r\sqrt{1-r^2}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\,r\,dr\,d\theta \quad u = 1-r^2,\ du = -2r\,dr,\ \text{so } -\frac{1}{2}du = r\,dr \\
&= \int_0^{2\pi}\int_1^0 \sqrt{u}\,(-\frac{1}{2}du)\,d\theta = \int_0^{2\pi}\frac{1}{2}\int_0^1 u^{1/2}\,du\,d\theta = \int_0^{2\pi}\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_0^1\,d\theta \\
&= \int_0^{2\pi}\frac{1}{3}\,d\theta = \frac{2\pi}{3}.
\end{aligned}$$

For spherical coordinates:

$$\begin{aligned}
\text{Vol} &= \int_0^{2\pi}\int_0^1\int_0^{\pi/2} \rho^2 \sin\phi\,d\phi\,d\rho\,d\theta \\
&= \int_0^{2\pi}\int_0^1 -\rho^2\cos\phi\Big|_0^{\pi/2}\,d\rho\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \rho^2\,d\rho\,d\theta \\
&= \int_0^{2\pi}\frac{1}{3}\rho^3\Big|_0^1\,d\theta \\
&= \int_0^{2\pi}\frac{1}{3}\,d\theta = \frac{2\pi}{3}.
\end{aligned}$$

8. [9 points] Let $f(x,y)$ be differentiable on $\mathbb{R}^2$.

(a) State the definition of the partial derivatives $f_x(0,0)$ and $f_y(0,0)$.

Solution: For the given function:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}.$$

(b) Suppose that $f_x(0,0) = 2$ and that the directional derivative of f at $(0,0)$ in the direction $\mathbf{u} = \frac{1}{\sqrt{2}}(1,1)$ is $5/\sqrt{2}$. Determine the value of $f_y(0,0)$.

Solution: By the formula for the directional derivative,

$$D_{\mathbf{u}}f(0,0) = \nabla f(0,0) \cdot \mathbf{u} = f_x(0,0)\,\frac{1}{\sqrt{2}} + f_y(0,0)\,\frac{1}{\sqrt{2}}$$

where $\mathbf{u} = \frac{1}{\sqrt{2}}(1,1) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. From the given conditions we know $D_{\mathbf{u}}f(0,0) = \frac{5}{\sqrt{2}}$ and $f_x(0,0) = 2$. Substituting these numbers into the above equation yields $f_y(0,0) = 3$.

9. [10 points] Let $f(x,y) = 4xy - x^4 - y^4$.

(a) Find the critical points of $f(x,y)$.

Solution: Since f is a polynomial, it is continuous. The critical points occur when

$$f_x(x,y) = 4y - 4x^3 = 0 \qquad f_y(x,y) = 4x - 4y^3 = 0.$$

Thus $y = x^3$ and $x = y^3$. Substituting the first equation into the second gives $x = (x^3)^3 = x^9$, so $0 = x^9 - x = x(x^8 - 1)$. Thus $x = 0$ or $x^8 = 1$, i.e., $x = \pm 1$. Since $y = x^3$, we obtain the solutions $x = y = 0$, $x = y = 1$ and $x = y = -1$ Therefore the critical points are $(0,0)$, $(1,1)$ and $(-1,-1)$.

(b) Use the second derivative test to classify the critical points as local maxima, local minima or saddle points.

Solution: First compute the second derivatives.

$$f_{xx}(x,y) = -12x^2, \qquad f_{xy}(x,y) = 4, \qquad f_{yy}(x,y) = -12y^2.$$
$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = 144x^2y^2 - 16.$$

Since $D(0,0) = -16 < 0$, it follows from the second derivative test that the origin is a saddle point.

Since $D(1,1) = 128 > 0$ and $f_{xx}(1,1) = -12 < 0$, $(1,1)$ is a local maximum.

Since $D(-1,-1) = 128 > 0$ and $f_{xx}(-1,-1) = -12 < 0$, $(-1,-1)$ is also a local maximum.

10. [10 points] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}8\\-1\end{pmatrix}$ and $T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-2\\3\end{pmatrix}$.

(a) Find the matrix of T with respect to the standard basis of $\mathbb{R}^2$.

Solution: Let $\mathbf{u} = \begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}1\\-1\end{pmatrix}$. Notice that

$$\mathbf{u} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \mathbf{e}_1 + \mathbf{e}_2 \qquad \mathbf{v} = \begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \mathbf{e}_1 - \mathbf{e}_2.$$

Therefore we have

$$\mathbf{e}_1 = \frac{1}{2}(\mathbf{u}+\mathbf{v}) \qquad \mathbf{e}_2 = \frac{1}{2}(\mathbf{u}-\mathbf{v}).$$

Since T is a linear transformation,

$$T(\mathbf{e}_1) = T\big(\frac{1}{2}(\mathbf{u}+\mathbf{v})\big) = \frac{1}{2}T(\mathbf{u}) + \frac{1}{2}T(\mathbf{v}) = \frac{1}{2}\begin{pmatrix}8\\-1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} = 3\mathbf{e}_1 + \mathbf{e}_2$$

$$T(\mathbf{e}_2) = T\big(\frac{1}{2}(\mathbf{u}-\mathbf{v})\big) = \frac{1}{2}T(\mathbf{u}) - \frac{1}{2}T(\mathbf{v}) = \frac{1}{2}\begin{pmatrix}8\\-1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}5\\-2\end{pmatrix} = 5\mathbf{e}_1 - 2\mathbf{e}_2.$$

So the matrix representation of T is $A = \begin{pmatrix}3 & 5\\1 & -2\end{pmatrix}$. Thus $T(\mathbf{w}) = A\mathbf{w}$ for $\mathbf{w} \in \mathbb{R}^2$.

(b) What is the rank of the matrix of part (a)? Is T one-to-one? Onto? Justify your answers.

Solution: The columns $\begin{pmatrix}3\\1\end{pmatrix}$ and $\begin{pmatrix}5\\-2\end{pmatrix}$ are clearly linearly independent, so the rank of the matrix A of part (a) is 2.

Since $T(\mathbf{v}) = A\mathbf{v}$, the range of T is the column space of A, and the dimension of the column space is the rank. Hence the range $R(T)$ of T has dimension 2. Since $R(T) \subseteq \mathbb{R}^2$, it follows that $R(T) = \mathbb{R}^2$. Thus T is onto.

The rank-nullity theorem implies that $\dim N(T) + \dim R(T) = \dim \mathbb{R}^2$, where $N(T) = \ker(T)$ is the nullspace or kernel of T. Hence $\dim N(T) = 0$, so that $N(T) = \{0\}$. This proves that T is one-to-one.

11. [10 points] Let $T : V \to W$ be linear. Also assume that $v_1, v_2, \ldots, v_k$ form a basis of the nullspace of T and that these vectors can be extended to a basis $v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n$ of V.

(a) Express the dimension of the range of T in terms of n and k.

Solution: According to the given information, the nullity of T is k and the dimension of V is n. Therefore by the rank-nullity theorem we get the rank of $T = n - k$, i.e. the dimension of the range of T is $n - k$.

(b) Prove that $T(v_{k+1}), \dots, T(v_n)$ are linearly independent in W.

Solution: To prove $T(v_{k+1}), \dots, T(v_n)$ are linearly independent in W, we prove that

$$a_{k+1}T(v_{k+1}) + a_{k+2}T(v_{k+2}) + \cdots + a_nT(v_n) = 0$$

implies

$$a_{k+1} = a_{k+2} = \cdots = a_n = 0.$$

Suppose that $a_{k+1}T(v_{k+1}) + a_{k+2}T(v_{k+2}) + \cdots + a_nT(v_n) = 0$. Because T is a linear transformation, it follows that

$$a_{k+1}T(v_{k+1}) + a_{k+2}T(v_{k+2}) + \cdots + a_nT(v_n) = T(a_{k+1}v_{k+1} + \cdots + a_nv_n) = 0.$$

Hence $a_{k+1}v_{k+1} + \cdots + a_nv_n$ is in the nullspace of T. Since $v_1, \dots, v_k$ forms a basis of the nullspace, there exist $b_1, \dots, b_k \in \mathbb{R}$ such that $a_{k+1}v_{k+1} + \cdots + a_nv_n = b_1v_1 + \cdots + b_kv_k$. Therefore we get the equation:

$$a_{k+1}v_{k+1} + \cdots + a_nv_n - b_1v_1 - \cdots - b_kv_k = 0.$$

Because $v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n$ form a basis of V, these vectors are linearly independent by definition. Hence

$$a_{k+1}v_{k+1} + \cdots + a_nv_n - b_1v_1 - \cdots - b_kv_k = 0$$

implies

$$a_{k+1} = a_{k+2} = \cdots = a_n = b_1 = b_2 = \cdots = b_k = 0.$$

Thereby we have shown

$$a_{k+1}T(v_{k+1}) + a_{k+2}T(v_{k+2}) + \cdots + a_nT(v_n) = 0$$

implies $a_{k+1} = a_{k+2} = \cdots = a_n = 0$. Hence $T(v_{k+1}), \dots, T(v_n)$ are linearly independent in W.

12. [10 points] Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Find an invertible matrix Q such that $Q^{-1}AQ$ is diagonal.

Solution: We first find all the eigenvalues of A, i.e., we find all possible $\lambda \in \mathbb{R}$ such that for each λ there exists $\mathbf{v} \neq 0$ and $A\mathbf{v} = \lambda\mathbf{v}$. To find such values, we find which λ would make $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 1 & 1-\lambda \end{bmatrix} = (-\lambda)(-\lambda)(1-\lambda) = 0, \text{ so } \lambda = 0 \text{ or } \lambda = 1.$$

If $\lambda = 0$, we find the nullspace of A:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to y + z = 0 \to y = -z.$$

Note that x and z are free variables. This gives the solutions

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -z \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

It follows that that for the eigenvalue $\lambda = 0$, the eigenspace E_0 (that is, the nullspace of $A - 0I$) has dimension 2 with a basis consisting of the eigenvectors

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{v_2} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

If $\lambda = 1$, we find the nullspace of $A - I$:

$$A - I = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{matrix} x - z = 0 \\ y = 0 \end{matrix} \to \begin{matrix} x = z \\ y = 0. \end{matrix}$$

The only free variable is z, giving the solutions

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus the eigenspace E_1 for $\lambda = 1$ is spanned by the single eigenvector

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore if we take $Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, we should have

$$Q^{-1}AQ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(One can check that $Q^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and that the above equation does hold.)