Substitution and Income Effects with the Cobb-Douglas

If $U(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ we know that

$$x_i = \frac{\alpha I}{p_i}$$
$$x_i^c = \alpha A^{-1} p_1^\alpha p_2^{1-\alpha} U$$
$$U = A p_1^{-\alpha} p_2^{(1-\alpha)} I$$

1. Own price Slutsky Equation: $\frac{\partial x_i}{\partial p_i} = \frac{\partial x_i^c}{\partial p_i} - x_i \frac{\partial x_i}{\partial I}$. Using the Marshallian demand function directly gives $\frac{\partial x_i}{\partial p_i} = -\frac{\alpha I}{p_i^2}$. Calculating the components of the Slutsky Equation gives:

$$\frac{\partial x_i^c}{\partial p_i} = \alpha (\alpha - 1) A^{-1} p_1^{\alpha-1} p_2^{1-\alpha} U = \alpha (\alpha - 1) p_i^{-2} I$$
$$-x_i \frac{\partial x_i}{\partial I} = -\frac{\alpha I}{p_i} \frac{\alpha}{p_i} = -\alpha^2 p_i^{-2} I$$

Summing these two effects gives $\frac{\partial x_i}{\partial p_i} = \text{Sub + Income Effect} = -\alpha p_i^{-2} I$ just as was derived from direct differentiation of the Marshallian demand function. Note, if $\alpha = 0.5$, we have

$$\frac{\partial x_i}{\partial p_i} = -0.5 p_i^{-2} I$$
$$\frac{\partial x_i^c}{\partial p_i} = -0.25 p_i^{-2} I$$
$$-x_i \frac{\partial x_i}{\partial I} = -0.25 p_i^{-2} I$$

That is, the total price effect in the Marshallian demand function is half income effect and half substitution effect. If, say, $\alpha = 0.3$, the substitution effect would be 70 percent of the total effect $(0.7 = \frac{0.3 \cdot 0.7}{0.3})$ and the income effect would be only 30 percent of the total. These proportions would be reversed if $\alpha = 0.7$. 
2. Cross-price Slutsky equation: \[
\frac{\partial x_1}{\partial p_2} = \frac{\partial x_i^c}{\partial p_2} - x_2 \frac{\partial x_i}{\partial I}.
\]

Now direct differentiation gives: \[
\frac{\partial x_1}{\partial p_2} = 0 \quad \text{and we wish to know why.}
\]

To calculate the Slutsky Equation we have to know the Marshallian demand for good 2 which is \[x_2 = \frac{(1-\alpha)I}{p_2} \] . So the Slutsky components are:

\[
\frac{\partial x_i^c}{\partial p_2} = \alpha(1-\alpha)A^{-1} p_i^{\alpha-1} p_2^{-\alpha} U = \alpha(1-\alpha) p_i^{-1} p_2^{-1} I \\
-x_2 \frac{\partial x_i}{\partial I} = -(1-\alpha) p_2^{-1} \cdot \alpha p_i^{-1} = -\alpha(1-\alpha) p_i^{-1} p_2^{-1} I
\]

Which shows that the substitution and income effects always precisely cancel out regardless of the value of \(\alpha\) . That makes the Cobb-Douglas a very special case.