Amherst College
Economics 58

The Cobb-Douglas Functions in Consumer Theory

Utility Function: $U(x,y) = x^{\alpha} y^{1-\alpha}$ **.**

Lagrangian: $L = x^{\alpha} y^{1-\alpha} + \lambda(I - p_x x - p_y y)$

FOCs:

$$\frac{\partial L}{\partial x} = \alpha x^{\alpha-1} y^{1-\alpha} - \lambda p_x = 0$$
$$\frac{\partial L}{\partial y} = (1-\alpha) x^{\alpha} y^{-\alpha} - \lambda p_y = 0$$
$$\frac{\partial L}{\partial \lambda} = I - p_x x - p_y y = 0$$

Solving these yields:

$$x = \frac{\alpha I}{p_x}$$
$$y = \frac{(1-\alpha) I}{p_y}$$

Indirect Utility: $V = x^{\alpha} y^{1-\alpha} = \dfrac{\alpha^{\alpha}(1-\alpha)^{1-\alpha} I}{p_x^{\alpha} p_y^{1-\alpha}} = KIp_x^{-\alpha} p_y^{\alpha-1}$**;** $K = \alpha^{\alpha}(1-\alpha)^{1-\alpha}$

Expenditure Function:

$$E = K^{-1} p_x^{\alpha} p_y^{1-\alpha} V$$
$$\ln E = -\ln K + \alpha \ln p_x + (1-\alpha)\ln p_y + \ln V$$

Numerical Examples: $\alpha = 0.5,\ x = .5I/p_x,\ y = .5I/p_y, V = .5I/\sqrt{p_x p_y},\ E = 2Vp_x^{.5} p_y^{.5}$

Lump Sum Principle: $I = 10,\ p_x = 1,\ p_y = 1, U = V = 5$

Two ways to get to $V = 6$

1. Just give this person $2.
2. Subsidize x. How much? Use $V = 6 = 5/\sqrt{p_x \cdot 1}$. So,

   $p_x = (5/6)^2 \approx 0.7$.

   Subsidize x at 0.3 per unit

At this price $x = 5/0.7 = 7.14$
Cost of subsidy = $0.3 \cdot 7.14 = 2.14$ -- costs more than
Income grant. Value of $1 in subsidy =2/2.14=0.93

Efficiency in Taxation: Govt. needs $2. Utility cost of income tax is 1; *V* falls from 5 to 4.

If impose a per unit tax of *t* on *x*, price becomes $p_x = 1 + t$.
$x = 5/(1+t), \quad x + tx = 5, \; but \; tx = 2 \quad so \; x = 3, \; t = 2/3$
$V = 5\big/\sqrt{5/3} = 3.87$. This per unit tax has an "excess burden" of 0.13 units of V.

To put a dollar cost on this, note that $.26 would compensate for the 0.13 units of V lost. Hence, the excess burden is 13 percent of the tax collected.

Exercise: Find excess burden of another dollar of tax. What is the marginal excess burden of the third dollar of tax?