Math 13 Spring 2010: Exam 1
February 23, 2010

Name:

Instructions: Each problem is scored out of 8 points for a total of 32 points. You may not use any outside materials(eg. notes or calculators). You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Problem 1.

(a) Find an equation for the tangent line to $\vec{r}(t) = \langle \cos t, 2\sin t, t\rangle$ at $(1, 0, 0)$.

(b) Find the distance from $(1, 4, 2)$ to the plane $2x - 6y + z = 3$.

(c) Find the angle of intersection of the two planes $2x + y - 3z = -1$ and $x + 2y - z = 4$.

Proof.

(a) The point occurs at $t = 0$. We have $\vec{r}\,'(t) = \langle -\sin t, 2\cos t, 1\rangle$ and hence $\vec{r}\,'(0) = \langle 0, 2, 1\rangle$, which is the direction of the line. We have the line

$$\langle 1, 2t, t\rangle\,.$$

(b) We use the distance formula

$$\begin{aligned} d &= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|2 - 24 + 2 - 3|}{\sqrt{4 + 36 + 1}} \\ &= \frac{23}{\sqrt{41}}. \end{aligned}$$

(c) We know the angle between two planes is the angle between their normal vectors. The two normal vectors are

$$\vec{n}_1 = \langle 2, 1, -3\rangle \quad \text{and} \quad \vec{n}_2 = \langle 1, 2, -1\rangle\,.$$

We compute the angle between them as

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \frac{2 + 2 + 3}{\sqrt{14}\sqrt{6}} = \frac{7}{\sqrt{84}}.$$

So we have

$$\theta = \cos^{-1}\frac{7}{\sqrt{84}}.$$

□

Problem 2.

(a) Find the vector function that represents the curve of intersection of the paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$. Find the point(s) where this curve intersects the plane $x + 2y = 3$?

(b) Suppose you start at the point $(0, 0, 3)$ and move 5π units along the curve $\vec{r}(t) = \langle 3 \sin t, 4t, 3 \cos t \rangle$. Where are you now?

Proof.

(a) We parameterize the cylinder as $x = t, y = t^2$. Then we use the paraboloid to get

$$\vec{r}(t) = \left\langle t, t^2, 4t^2 + t^4 \right\rangle.$$

To find the point(s) of intersection we solve

$$t + 2t^2 - 3 = (t - 1)(2t + 3) = 0$$

to get $t = 1, t = -\frac{3}{2}$ to get the two points

$$\vec{r}(1) = \langle 1, 1, 5 \rangle \quad \text{and} \quad \vec{r}\left(-\frac{3}{2}\right) = \left\langle -\frac{3}{2}, \frac{9}{4}, 9 + \frac{81}{16} \right\rangle.$$

(b) We parameterize with respect to arc length. We compute

$$\vec{r}\,'(t) = \langle 3 \cos t, 4, -3 \sin t \rangle$$
$$|\vec{r}\,'(t)| = \sqrt{9 \cos^2 t + 16 + 9 \sin^2 t} = \sqrt{25} = 5.$$

The point $(0, 0, 3)$ occurs at $t = 0$ so we have

$$\begin{aligned} s(t) &= \int_0^t |\vec{r}\,'(t)|\, dt \\ &= \int_0^t 5dt = 5t \end{aligned}$$

So we have $t = \frac{s}{5}$. So the parametrization with respect to arc length is

$$r(s) = \left\langle 3 \sin \frac{s}{5}, \frac{4}{5}s, 3 \cos \frac{s}{5} \right\rangle;$$

So we have

$$r(5\pi) = \langle 0, 4\pi, -3 \rangle.$$

□

Problem 3. Given the curve $\vec{r}(t) = \left\langle t^2 - 1, 2t + 3, t^2 - 4t \right\rangle$

(a) Find the curvature at $(3, 7, -4)$.

(b) Find an equation for the normal plane at $(3, 7, -4)$.

Proof.

(a) The point occurs at $t = 2$. We compute

$$\begin{aligned} \vec{r}\,'(t) &= \langle 2t, 2, 2t - 4 \rangle \\ \vec{r}\,'(2) &= \langle 4, 2, 0 \rangle \\ |\vec{r}\,'(2)| &= \sqrt{20} \\ \vec{r}\,''(t) &= \langle 2, 0, 2 \rangle \end{aligned}$$

We compute

$$\langle 4, 2, 0 \rangle \times \langle 2, 0, 2 \rangle = \langle 4, -8, 4 \rangle = 4 \langle 1, -2, -1 \rangle$$

So we have

$$\kappa = \frac{|4 \langle 1, -2, -1 \rangle|}{|\langle 4, 2, 0 \rangle|^3} = \frac{4\sqrt{6}}{20\sqrt{20}} = \frac{\sqrt{3}}{5\sqrt{10}}.$$

(b) The normal plane has normal vector $\vec{T}$. So compute

$$\begin{aligned} \vec{T}(t) &= \frac{1}{\sqrt{4t^2 + 4 + (2t - 4)^2}} \langle 2t, 2, 2t - 4 \rangle \\ \vec{T}(2) &= \frac{1}{\sqrt{20}} \langle 4, 2, 0 \rangle \end{aligned}$$

which is in the direction $\langle 2, 1, 0 \rangle$. So we have the plane

$$2(x - 3) + (y - 7) = 0.$$

□

Problem 4. Given $\vec{r}(t) = 2t^2\hat{i} + t\hat{j} + \hat{k}$.

(a) Find the tangential and normal components of acceleration as functions of t.

(b) What is the minimum speed and for what t does the minimum speed occur?

Proof.

(a) We have

$$r'(t) = \langle 4t, 1, 0\rangle$$
$$r''(t) = \langle 4, 0, 0\rangle$$

so we have

$$a_T = \frac{r' \cdot r''}{|r'|} = \frac{16t}{\sqrt{16t^2+1}}$$
$$a_N = \frac{|r' \times r''|}{|r'|} = \frac{|0, 0, -4|}{\sqrt{16t^2+1}} = \frac{4}{\sqrt{16t^2+1}}.$$

(b) We compute speed as

$$v = |\vec{r}'(t)| = \sqrt{16t^2+1}.$$

This is a max/min problem so we solve for the critical points:

$$\frac{d}{dt}\vec{v}(t) = a_T.$$

$$a_T = \frac{16t}{\sqrt{16t^2+1}} = 0$$

to get $t = 0$. This gives a minimum speed of 1 and is clearly a minimum since v contains only t^2.

$\square$