Insert 13.3 Long Insert representing most of remainder of chapter.

A MATHEMATICAL MODEL OF EXCHANGE

Although the prior graphical model of general equilibrium with two goods is fairly instructive, it cannot reflect all of the features of general equilibrium modeling with an arbitrary number of goods and productive inputs. In the remainder of this chapter we will illustrate how such a more general model can be constructed and look at some of the insights that such a model can provide. For most of our presentation we will look only at a model of exchange – quantities of various goods already exist and are merely traded among individuals. In such a model there is no production. Later in the chapter we will look briefly at how production can be incorporated into the general model we have constructed.

Vector Notation

Most general equilibrium modeling is conducted using vector notation. This provides great flexibility in specifying an arbitrary number of goods or individuals in the models. Consequently, this seems to be a good place to offer a brief introduction to such notation. A "vector" is simply an ordered array of variables (which each may take on specific values). Here we will usually adopt the convention that the vectors we use are column vectors. Hence, we will write an *n* x 1 column vector as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \qquad \textbf{(3.21)}$$

where each x_i is a variable that can take on any value. If **x** and **y** are two *n* x 1 column vectors, then the (vector) sum of them is defined as:

$$\mathbf{x}+\mathbf{y}=\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}+\begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix}=\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ . \\ . \\ . \\ x_n+y_n \end{bmatrix} \qquad \textbf{(3.22)}$$

Notice that this sum only is defined if the two vectors are of equal length. In fact, checking the length of vectors is one good way of deciding whether one has written a meaningful vector equation or not.

The (dot) product of two vectors is defined as the sum of the component-by-component product of the elements in the two vectors. That is:

$$\mathbf{xy}=x_1y_1+x_2y_2+...+x_ny_n \qquad \textbf{(13.23)}$$

Notice again that this operation is only defined if the vectors are of the same length. With these few concepts we are now ready to illustrate the general equilibrium model of exchange.

Utility, initial endowments and budget constraints

In our model of exchange there are assumed to be *n* goods and *m* individuals. Each individual gains utility from the vector of goods he or she consumes $u^i(\mathbf{x}^i)$ where $i=1...m$**.** Individuals also possess initial endowments of the goods given by $\bar{\mathbf{x}}^i$. Individuals are free to exchange their initial endowments with other individuals or to keep some or all of the endowment for themselves. In their trading individuals are assumed to be price-takers – that is, they face a price vector (**p)** that specifies the market price for each of the n goods. Each individual seeks to maximize utility and is bound by a budget constraint that requires that the total amount spent on consumption equals the total value of his or her endowment:

$$\mathbf{p}\mathbf{x}^i = \mathbf{p}\overline{\mathbf{x}}^i \qquad \textbf{(13.24)}$$

Although this budget constraint has a very simple form, it may be worth contemplating it for a minute. The right side of Equation 13.24 is the market value of this individual's endowment (sometimes referred to as his or her "full income"). He or she could "afford" to consume this endowment (and only this endowment) if he or she wished to be self-sufficient. But the endowment can also be spent on some other consumption bundle (which, presumably, provides more utility). Because consuming items in one's own endowment has an opportunity cost, the terms on the left of Equation 13.24 consider the costs of all items that enter into the final consumption bundle, including endowment goods that are retained.

Demand functions and homogeneity

The utility maximization problem outlined in the previous section is identical to the one we studied in detail in Part 2 of this book. As we showed in Chapter 4, one outcome of this process is a set of n individual demand functions (one for each good) in which quantities demanded depend on all prices and income. Here we can denote these in vector form as $\mathbf{x}^i(\mathbf{p}, \mathbf{p}\overline{\mathbf{x}}^i)$. These demand functions are continuous and, as we showed in Chapter 4, they are homogeneous of degree zero in all prices and income. This latter property can be indicated in vector notation by

$$\mathbf{x}^i(t\mathbf{p}, t\mathbf{p}\overline{\mathbf{x}}^i) = \mathbf{x}^i(\mathbf{p}, \mathbf{p}\overline{\mathbf{x}}^i) \qquad \textbf{(13.25)}$$

for any $t > 0$. This property will be quite useful because it will permit us to adopt a convenient normalization scheme for prices which, because it does not alter relative prices, leaves quantities demanded unchanged.

Equilibrium and Walras Law

Equilibrium in this simple model of exchange requires that the total quantities of each good demanded be equal to the total endowment of each good available (remember, there is no production in this model). Because the model used is very similar to the one originally developed by Leon Walras[9], this equilibrium concept is customarily attributed to him.

Definition

Walrasian Equilibrium A Walrasian Equilibrium is an allocation of resources and an associated price vector, $\mathbf{p}^*$, such that

$$\sum_{i=1}^{m} \mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^*\overline{\mathbf{x}}^i) = \sum_{i=1}^{m} \overline{\mathbf{x}}^i \qquad \textbf{(13.26)}$$

where the summation is taken over the *m* individuals in this exchange economy.

The *n* equations in Equation 13.26 just state that in equilibrium demand equals supply in each market. This is the multi-market analogue of the single market equilibria examined in the previous chapter. Since there are *n* prices to be determined, a simple counting of equations and unknowns might suggest that the existence of such a set of prices is

[9] The concept is named for the Nineteenth Century French/Swiss economist Leon Walras who pioneered the development of general equilibrium models. Models of the type discussed in this chapter are often referred to as models of "Walrasian Equilibrium" primarily because of the price-taking assumptions inherent in them.

guaranteed by the simultaneous equation solution procedures studied in elementary algebra. Such a supposition would be incorrect for two reasons. First, the algebraic theorem about simultaneous equations systems applies only to linear equations. Nothing suggests that the demand equations in this problem will be linear – in fact, most of the examples of demand equations we encountered in Part 2 were definitely non-linear.

A second problem with the equations in 13.26 is that they are not independent of one another – they are related by what is known as "Walras Law". Because each individual in this exchange economy is bound by a budget constraint of the form given in Equation 13.24, we can sum over all individuals to obtain

$$\sum_{i=1}^{m} \mathbf{p}\mathbf{x}^i = \sum_{i=1}^{m} \mathbf{p}\overline{\mathbf{x}}^i \quad \text{or} \quad \sum_{i=1}^{m} \mathbf{p}(\mathbf{x}^i - \overline{\mathbf{x}}^i) = 0 \qquad \textbf{(13.27)}$$

In words, Walras Law states that the value of all quantities demanded must equal the value of all endowments. This result holds for any set of prices, not just for equilibrium prices[10]. The general lesson is that the logic of individual budget constraints necessarily creates a relationship among the prices in any economy. It is this connection that helps to ensure that a demand/supply equilibrium exists, as we now show.

Existence of equilibrium in the exchange model

The question of whether all markets can reach equilibrium together has fascinated economists for nearly 200 years. Although intuitive evidence from the real world suggests that this must indeed be possible (market prices do not tend to fluctuate wildly from one day to the next), proving the result mathematically proved to be rather difficult. Walras himself thought he had a good proof that relied on evidence from the market to adjust prices toward equilibrium. The price would rise for any good for which demand

[10] Walras Law holds trivially for equilibrium prices as multiplication of Equation 13.26 by **p** shows.

exceeded supply and fall when supply exceeded demand. Walras believed that if this process continued long enough, a full set of equilibrium prices would eventually be found. Unfortunately, the pure mathematics of Walras' solution were difficult to state and, ultimately, there was no guarantee that a solution would be found. But Walras' idea of adjusting prices toward equilibrium using market forces provided a starting point for the modern proofs, which were largely developed during the 1950s.

A key aspect of the modern proofs of the existence of equilibrium prices is the choice of a good normalization rule. Homogeneity of demand functions makes it possible to use any absolute scale for prices, providing that relative prices are unaffected by this choice. An especially convenient such scale is to normalize prices so that they sum to one. Consider an arbitrary set of *n* non negative prices $p_1, p_2 ... p_n$. We can normalize[11] these to form a new set of prices

$$p_i^{'} = \frac{p_i}{\sum_{k=1}^{n} p_k} \qquad \textbf{(13.28)}$$

These new prices will have the properties that $\sum_{k=1}^{n} p_k^{'} = 1$ and that relative price ratios are maintained:

$$\frac{p_i^{'}}{p_j^{'}} = \frac{p_i / \sum p_k}{p_j / \sum p_k} = \frac{p_i}{p_j} \qquad \textbf{(13.29)}$$

Because this sort of mathematical process can always be done, we will assume, without loss of generality, that the price vectors we use ($\mathbf{p}$) have all been normalized in this way.

[11] This is possible only if at least one of the prices is non-zero. Throughout our discussion we will assume that not all equilibrium prices can be zero.

Proving the existence of equilibrium prices in our model of exchange therefore amounts to showing that there will always exist a price vector, $\mathbf{p}^*$ that achieves equilibrium in all markets. That is

$$\sum_{i=1}^{m}\mathbf{x}^i(\mathbf{p}^*,\mathbf{p}^*\overline{\mathbf{x}}^i)=\sum_{i=1}^{m}\overline{\mathbf{x}}^i \text{ or } \sum_{i=1}^{m}\mathbf{x}^i(\mathbf{p}^*,\mathbf{p}^*\overline{\mathbf{x}}^i)-\sum_{i=1}^{m}\overline{\mathbf{x}}^i=\mathbf{0} \text{ or } \mathbf{z}(\mathbf{p}^*)=\mathbf{0} \quad \textbf{(13.30)}$$

where we use $\mathbf{z}(\mathbf{p})$ as a shorthand way of recording the "excess demands" for goods at a particular set of prices. In equilibrium, excess demand is zero in all markets[12].

Now consider the following way of implementing Walras' idea that goods in excess demand should have their prices increased whereas those in excess supply should have their prices reduced[13]. Starting from any arbitrary set of prices, $\mathbf{p}_0$, we define a new set, $\mathbf{p}_1$, as

$$\mathbf{p}_1=f(\mathbf{p}_0)=\mathbf{p}_0+k\,\mathbf{z}(\mathbf{p}_0)\,. \quad \textbf{(13.31)}$$

where k is a positive constant. This function will be continuous (because demand functions are continuous) and it will map one set of normalized prices into another (because of our assumption that all prices are normalized). Hence it will meet the conditions of the Brouwer Fixed Point Theorem which states that any continuous function from a closed compact set onto itself (in the present case, from the "unit simplex" onto itself) will have a "fixed point" such that $\mathbf{x}=f(\mathbf{x})$. The Theorem is illustrated for a single dimension in Figure 13.7. There, no matter what shape the

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[12] Goods that are in excess supply at equilibrium will have a zero price. We will not be concerned with such "free goods" here.

[13] What follows is an extremely simplified version of the proof of the existence of equilibrium prices. In particular, problems of free goods and appropriate normalizations have been largely assumed away. For a mathematically correct proof, see, for example, G. Debreu, *Theory of Value* (New York, John Wiley & Sons, 1959).

function $f(x)$ takes, so long as it is continuous, it must somewhere cross the 45^0 line and at that point $x = f(x)$.

[old Figure 13.7 goes here]

If we let $\mathbf{p}^*$ represent the fixed point identified by the Brouwer Theorem for Equation 13.31, we have:

$$\mathbf{p}^* = f(\mathbf{p}^*) = \mathbf{p}^* + k\,\mathbf{z}(\mathbf{p}^*)\ . \qquad \textbf{(13.32)}$$

Hence, at this point, $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$, so $\mathbf{p}^*$ is an equilibrium price vector. The proof that Walras sought is easily accomplished using an important mathematical result developed a few years after his death. The elegance of the proof may obscure the fact that it uses a number of assumptions about economic behavior such as: (1) Price taking by all parties; (2) Homogeneity of demand functions; (3) Continuity of demand functions; and (4) Presence of budget constraints and Walras Law. All of these play important roles in showing that a system of simple markets can indeed achieve a multi-market equilibrium.

First theorem of welfare economics

Given that the forces of supply and demand can establish equilibrium prices in the general equilibrium model of exchange we have developed, it is natural to ask what the welfare consequences of this finding are. Adam Smith[14] hypothesized that market forces provide an "invisible hand" that leads each market participant to "promote an end [social welfare] which was no part of his intention." Modern welfare economics seeks to understand the extent to which Smith was correct.

Perhaps the most important welfare result that can be derived from the exchange model is that the resulting Walrasian Equilibrium is "efficient" in the sense that it is not

[14] Adam Smith *The Wealth of Nations* (New York, Modern Library, 1937) page 423.

possible to devise some alternative allocation of resources in which at least some people are better off and no one is worse off. This definition of efficiency was originally developed by Italian economist Vilfredo Pareto in the early 1900's. Understanding the definition is easiest if we consider what an "inefficient" allocation might be. The total quantities of goods included in initial endowments would be allocated inefficiently if it were possible, by shifting goods around among individuals, to make at least one person better off (that is, receive a higher utility) and no one worse off. Clearly, if individuals' preferences are to count, such a situation would be undesirable. Hence, we have a formal definition,

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Definition

Pareto Efficient Allocation An allocation of the available goods in an exchange economy is efficient if it is not possible to devise an alternative allocation in which at least one person is better off and no one is worse off.

A proof that all Walrasian Equilibria are Pareto Efficient proceeds indirectly. Suppose that $\mathbf{p}^*$ generates a Walrasian equilibrium in which the quantities of goods consumed by each person is denoted by ${}^*\mathbf{x}^k \quad (k = 1...m)$. Now assume that there is some alternative allocation of the available goods ${}^{'}\mathbf{x}^k \quad (k = 1...m)$ such that, for at least one person, say person *i*, it is that case that ${}^{'}\mathbf{x}^i$ is preferred to ${}^*\mathbf{x}^i$. For this person, it must be the case that

$$\mathbf{p}^* \, {}^{'}\mathbf{x}^i > \mathbf{p}^* \, {}^*\mathbf{x}^i \qquad \textbf{(13.33)}$$

because otherwise this person would have bought the preferred bundle in the first place. If all other individuals are to be equally well off under this new proposed allocation it must be the case for them that

$$\mathbf{p}^{*}\,'\mathbf{x}^{k} = \mathbf{p}^{*}\,{}^{*}\mathbf{x}^{k} \quad k = 1...m, k \neq i\ . \qquad \textbf{(13.34)}$$

If the new bundle were less expensive, such individuals could not have been minimizing expenditures at $\mathbf{p}^{*}$. Finally, in order to be feasible, the new allocation must obey the quantity constraints

$$\sum_{i=1}^{m}{}'\mathbf{x}^{i} = \sum_{i=1}^{m}\overline{\mathbf{x}}^{i}\ . \qquad \textbf{(13.35)}$$

Multiplying Equation 13.35 by $\mathbf{p}^{*}$ yields

$$\sum_{i=1}^{m}\mathbf{p}^{*}\,'\mathbf{x}^{i} = \sum_{i=1}^{m}\mathbf{p}^{*}\,\overline{\mathbf{x}}^{i} \qquad \textbf{(13.36)}$$

but Equations 13.33 and 13.34 together with Walras' Law applied to the original equilibrium imply that

$$\sum_{i=1}^{m}\mathbf{p}^{*}\,'\mathbf{x}^{i} > \sum_{i=1}^{m}\mathbf{p}^{*}\,{}^{*}\mathbf{x}^{i} = \sum_{i=1}^{m}\mathbf{p}^{*}\,\overline{\mathbf{x}}^{i}\ . \qquad \textbf{(13.37)}$$

Hence we have a contradiction and must conclude that no such alternative allocation can exist. We can therefore summarize our analysis by

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Definition

First Theorem of Welfare Economics Every Walrasian Equilibrium is Pareto Efficient.

The significance of this "theorem" should not be overstated. The theorem does not say that every Walrasian Equilibrium is in some sense socially desirable. Walrasian

Equilibria can, for example, exhibit vast inequalities among individuals arising in part from inequalities in their initial endowments (see the discussion in the next section). The theorem also assumes price-taking behavior and full information about prices – assumptions that need not hold in other models. Finally, the theorem does not consider possible effects of one individual's consumption on another. In the presence of such externalities even a perfect competitive price system may not yield Pareto optimal results (see Chapter 19).

Still, the theorem does show that Smith's "invisible hand" conjecture has some validity. The simple markets in this exchange world can find equilibrium prices and at those equilibrium prices the resulting allocation of resources will be efficient in the Pareto sense. Developing this proof is one of the key achievements of welfare economics.

A graphic illustration of the first theorem

In Figure 13.8 we again employ the Edgeworth Box diagram, this time to illustrate an exchange economy. In this economy there are only two goods (*x* and *y*) and two individuals (*A* and *B*). The total dimensions of the Edgeworth Box are determined by the total quantities of the two goods available ($\overline{x}$ and $\overline{\mathrm{y}}$). Goods allocated to individual *A* are recorded using 0_A as an origin. Individual *B* gets those quantities of the two goods that are "left over" and can be measured using 0_B as an origin. Individual *A*'s indifference curve map is drawn in the usual way, whereas individual *B*'s map is drawn from the perspective of 0_B . Point *E* in the Edgeworth Box represents the initial endowments of these two individuals. Individual *A* starts with $\overline{x}^A$ and $\overline{y}^A$. Individual *B* starts with $\overline{x}^B = \overline{\mathrm{x}} - \overline{x}^A$ and $\overline{y}^B = \overline{y} - \overline{y}^A$.

[New Figure 13.8 goes here]

The initial endowments provide a utility level of U_A^2 for person *A* and U_B^2 for person *B*. These levels are clearly inefficient in the Pareto sense. For example, we could, by reallocating the available goods, increase person *B*'s utility to U_B^3 while holding person *A*'s utility constant at U_A^2 (point[15] *B*). Or we could increase person *A*'s utility to U_A^3 while keeping person *B* on the U_B^2 indifference curve (point *A*). Allocations *A* and *B* are Pareto efficient, however, because at these allocations it is not possible to make either person better off without making the other worse off. There are many other efficient allocations in the Edgeworth box diagram. These are identified by the tangencies of the two individuals' indifference curves. The set of all such efficient points is shown by the line joining O_A to O_B. This line is sometimes called the "contract curve" because it represents all of the Pareto efficient contracts that might be reached by these two individuals. Notice, however, that (assuming that no individual would voluntarily opt for a contract that made him or her worse off) only contracts between points *B* and *A* are viable with initial endowments given by point *E*.

The line *PP* in Figure 13.8 shows the competitively established price ratio that is guaranteed by our earlier existence proof. The line passes through the initial endowments (*E*) and shows the terms at which these two individuals can trade away from these initial positions. Notice that such trading is beneficial to both parties – that is, it allows them to get a higher utility level than is provided by their initial endowments. Such trading will continue until all such mutual beneficial trades have been completed. That will occur at allocation E^* on the contract curve. Because the individuals' indifference curves are

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[15] This point could in principle be found by solving the following constrained optimization problem:
Maximize $U_B(x_B, y_B)$ subject to the constraint $U_A(x_A, y_A) = U_A^2$. See Example 13.3.

tangent at this point, no further trading would yield gains to both parties. The competitive allocation E^* therefore meets the Pareto criterion for efficiency, as we showed mathematically earlier.

Second Theorem of Welfare Economics

The first theorem of welfare economics shows that a Walrasian equilibrium is Pareto efficient, but the social welfare consequences of this result are limited because of the role played by initial endowments in the demonstration. The location of the Walrasian equilibrium at E^* in Figure 13.8 was significantly influenced by the designation of E as the starting point for trading. Points on the contract curve outside the range of AB are not attainable through voluntary transactions, even though these may in fact be more socially desirable than E^* (perhaps because utilities are more equal). The "second theorem" of welfare economics addresses this issue. It states that for any Pareto optimal allocation of resources there exists a set of initial endowments and a related price vector such that this allocation is also a Walrasian equilibrium. Phrased another way, any Pareto optimal allocation of resources can also be a Walrasian equilibrium providing that initial endowments are adjusted accordingly.

A graphical proof of the second theorem should suffice. Figure 13.9 repeats the key aspects of the exchange economy pictures in Figure 13.8. Given the initial endowments at point E, all voluntary Walrasian equilibrium must lie between points A and B on the contract curve. Suppose, however, that these allocations were thought to be undesirable – perhaps because they involve too must inequality of utility. Assume that the Pareto optimal allocation Q^* is believed to be socially preferable, but it is not attainable from the initial endowments at point E. The second theorem states that one can draw a price line through Q^* that is tangent to both individuals' respective indifference curves. This line is denoted by *P'P'* in Figure 13.9. Because the slope of this line shows potential trades these individuals are willing to make, any point on the line can serve as an initial endowment from which trades lead to Q^*. One such point is denoted by $\bar{Q}$. If a benevolent government wished to ensure that Q^* would emerge as a Walrasian equilibrium, it would have to transfer initial endowments of the goods from *E* to $\bar{Q}$ (making person *A* better off and person *B* worse off in the process).

[Figure 13.9 goes here]

Example 13.3 A Two-Person Exchange Economy

To illustrate these various principles, consider a simple two-person, two good exchange economy. Suppose that total quantities of the goods are fixed at $\bar{x} = \bar{y} = 1,000$. Person *A*'s utility takes the Cobb-Douglas form:

$$U_A(x_A, y_A) = x_A^{2/3} y_A^{1/3} \quad \textbf{(13.38)}$$

and person *B*'s preferences are given by:

$$U_B(x_B, y_B) = x_B^{1/3} y_B^{2/3} \quad \textbf{(13.39)}$$

Notice that person *A* has a relative preference for good *x* and person *B* has a relative preference for good *y*. Hence you might expect that the Pareto efficient allocations in this model would have the property that person *A* would consume relatively more *x* and person *B* would consume relatively more *y*. To find these allocations explicitly we need to find a way of dividing up the available goods in such a way that the utility of person *A* is maximized for any pre-assigned utility level for person *B*. Setting up the Lagrangian expression for this problem, we have:

$$\mathcal{L}(x_A, y_A) = U_A(x_A, y_A) + \lambda[U_B(1000 - x_A, 1000 - y_A) - \bar{U}_B] \qquad \textbf{(13.40)}$$

Substituting for the explicit utility functions assumed here yields

$$\mathcal{L}(x_A, y_A) = x_A^{2/3} y_A^{1/3} + \lambda[(1000 - x_A)^{1/3}(1000 - y_A)^{2/3} - \bar{U}_B] \qquad \textbf{(13.41)}$$

and the first order conditions for a maximum are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_A} &= \frac{2}{3}\left(\frac{y_A}{x_A}\right)^{1/3} - \frac{\lambda}{3}\left(\frac{1000 - y_A}{1000 - x_A}\right)^{2/3} = 0 \\ \frac{\partial \mathcal{L}}{\partial y_A} &= \frac{1}{3}\left(\frac{x_A}{y_A}\right)^{2/3} - \frac{2\lambda}{3}\left(\frac{1000 - x_A}{1000 - y_A}\right)^{1/3} = 0 \end{aligned} \qquad \textbf{(13.42)}$$

Moving the terms in λ to the right and dividing the top equation by the bottom gives

$$2\left(\frac{y_A}{x_A}\right) = \frac{1}{2}\left(\frac{1000 - y_A}{1000 - x_A}\right)$$

or $\qquad$ **(13.43)**

$$\frac{x_A}{1000 - x_A} = \frac{4y_A}{1000 - y_A}$$

This equation allows us to identify all of the Pareto optimal allocations in this exchange economy. For example, if we were to arbitrarily choose $x_A = x_B = 500$ Equation 13.43 would become

$$\frac{4y_A}{1000 - y_A} = 1 \quad \text{so} \quad y_A = 200,\ y_B = 800 \qquad \textbf{(13.44)}$$

This allocation is relatively favorable to person *B*. At this point on the contract curve $U_A = 500^{2/3} 200^{1/3} = 369, \quad U_B = 500^{1/3} 800^{2/3} = 683$. Notice that although the available quantity of *x* is divided evenly (by assumption), most of good *y* goes to person *B* as efficiency requires.

Equilibrium price ratio. In order to calculate the equilibrium price ratio at this point on the contract curve, we need to know the two individuals' marginal rates of substitution. For person *A*,

$$MRS = \frac{\partial U_A / \partial x_A}{\partial U_A / \partial y_A} = 2\frac{y_A}{x_A} = 2\frac{200}{500} = 0.8 \qquad \textbf{(13.45)}$$

and for person *B*

$$MRS = \frac{\partial U_B / \partial x_B}{\partial U_B / \partial y_B} = 0.5\frac{y_A}{x_A} = 0.5\frac{800}{500} = 0.8 \qquad \textbf{(13.46)}$$

Hence the marginal rates of substitution are indeed equal (as they should be) and they imply a price ratio of $p_x / p_y = 0.8$.

Initial endowments. Because this equilibrium price ratio will permit these individuals to trade 8 units of *y* for each 10 units of *x*, it is a simple matter to devise initial endowments consistent with this Pareto optimum. Consider, for example, the endowment $\bar{x}_A = 350,\ \bar{y}_A = 320; \bar{x}_B = 650,\ \bar{y}_B = 680$. If $p_x = 0.8,\ p_y = 1$, the value of person *A*'s initial endowment is 600. If he or she spends 2/3 of this amount on good *x*, it is possible to purchase 500 units of good *x* and 200 units of good *y*. This would raise utility from $U_A = 350^{2/3}320^{1/3} = 340$ to 369. Similarly, the value of person *B*'s endowment is 1200. If he or she spends 1/3 of this on good *x*, 500 units can be bought. With the remaining 2/3 of the value of the endowment being spent on good *y*, 800 units can be bought. In the process *B*'s utility rises from 670 to 683. So trading from the proposed initial endowment to the contract curve is indeed mutually beneficial (as shown in Figure 13.8).

QUERY: Why did starting with the assumption that good *x* would be divided equally on the contract curve result in a situation favoring person *B* throughout this problem? What point on the contract curve would provide equal utility to persons *A* and *B*? What would the price ratio of the two goods be at this point?

Social welfare functions

Figure 13.9 shows that there are many Pareto efficient allocations of the available goods in an exchange economy. We are assured by the second theorem of welfare economics that any of these can be supported by a Walrasian system of competitively determined prices, providing that initial endowments are adjusted accordingly. A major question for welfare economics, then, is how (if at all) to develop criteria for choosing among all of these allocations. In this section we look briefly at one strand of this very large topic – the study of "social welfare functions". Simply put, a social welfare function in a hypothetical scheme for ranking potential allocations of resources based on the utility they provide to individuals. In mathematical terms:

$$\textit{Social Welfare} = SW[U_1(\mathbf{x}^1), U_2(\mathbf{x}^2), ..., U_m(\mathbf{x}^m)] \qquad \textbf{(13.47)}$$

The "social planner's" goal then is to choose allocations of goods among the *m* individuals in the economy in a way that maximizes *SW*. Of course, this exercise is a purely conceptual one – in reality there are no clearly articulated social welfare functions in any economy and there are serious doubts about whether such a function could ever arise from some type of democratic process[16]. Still, assuming the existence of such a function can help to illuminate many of the thorniest problems in welfare economics.

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[16] The "impossibility" of developing a social welfare function from the underlying preferences of people in society was first studied by K. Arrow in *Social Choice and Individual Values* 2nd Edition (New York, Wiley, 1963). There is a very large literature stemming from Arrow's initial discovery.

A first observation that might be made about the social welfare function in Equation 13.47 is that any welfare maximum must also be Pareto efficient. If we assume that every individual's utility is to "count", it seems clear that any allocation that permits further Pareto improvements (that make one person better off and no one else worse off) cannot be a welfare maximum. Hence, achieving a welfare maximum is indeed a problem in choosing among Pareto efficient allocations and their related Walrasian price systems.

We can make further progress in examining the idea of social welfare maximization by considering the precise functional form that *SW* might take. Specifically, if we assume utility is measurable, using the CES form can be particularly instructive:

$$SW(U_1, U_2, ..., U_m) = \frac{U_1^R}{R} + \frac{U_2^R}{R} + ... + \frac{U_m^R}{R} \quad R \le 1. \qquad \textbf{(13.48)}$$

Because we have used this functional form many times before in this book, its properties should by now be familiar. Specifically, if $R = 1$, the function becomes:

$$SW(U_1, U_2, ..., U_m) = U_1 + U_2 + ... + U_m . \qquad \textbf{(13.49)}$$

So, utility is just a simple sum of the utility of every person in the economy. Such a social welfare function is sometimes called a "utilitarian" function. With such a function social welfare is judged by the aggregate sum of utility (or perhaps even income) with no regard for how utility (income) is distributed among the members of society.

At the other extreme, consider the case $R = -\infty$. In this case, social welfare has a "fixed proportions" character and (as we have seen in many other applications),

$$SW(U_1, U_2, ..., U_m) = Min[U_1, U_2, ..., U_m] \quad \textbf{(13.50)}$$

This function therefore focuses on the worse-off person in any allocation and chooses that allocation for which this person has the highest utility. Such a social welfare function is called a “maximin” function. If was made popular by the philosopher John Rawls who argued that if individuals did not know which position they would ultimately have in society (that is, they operate under a “veil of ignorance”), they would opt for this sort of social welfare function to guard against being the worst-off person[17]. Our analysis in Chapter 7 suggests that people may not be quite this risk averse in choosing social arrangements. But, Rawls’ focus on the bottom of the utility distribution is probably a good antidote to thinking about social welfare in purely utilitarian terms.

It is possible to explore many other potential functional forms for a hypothetical welfare function. Problem 13.14 looks at some connections between social welfare functions and the income distribution, for example. But such illustrations largely miss a crucial point if they focus only on an exchange economy. Because the quantities of goods in such an economy are fixed, issues related to production incentives do not arise when evaluating social welfare alternatives. In actuality, however, any attempt to redistribute income (or utility) through taxes and transfers will necessarily affect production incentives and therefore affect the size of the Edgeworth Box. Assessing social welfare will therefore involve studying the tradeoff between achieving distributional goals and maintaining levels of production. To examine such possibilities we must introduce production into our general equilibrium framework.

[17] J. Rawls *A Theory of Justice* (Cambridge, MA. Harvard University Press, 1971) .

A MATHEMATICAL MODEL OF PRODUCTION AND EXCHANGE

Adding production to the model of exchange developed in the previous section is a relatively simple process. First, the notion of a "good" needs to be expanded to include factors of production. So we will assume that our list of *n* goods now includes inputs also whose prices will be determined within the general equilibrium model. Some inputs for one firm in a general equilibrium model are produced by other firms. Some of these goods may also be consumed by individuals (cars are used by both firms and final consumers) and some of these may be used only as intermediate goods (steel sheets are used only to make cars and are not bought by consumers). Other inputs may be part of individuals' initial endowments. Most importantly this is the way labor supply is treated in general equilibrium models. Individuals are endowed with a certain number of potential labor hours. They may sell these to firms by taking jobs at competitively determined wages or they may choose to consume the hours themselves in the form of "leisure". In making such choices we continue to assume that individuals maximize utility[18].

We will assume that there are *r* firms involved in production. Each of these firms is bound by a production function that describes the physical constraints on the ways the firm can turn inputs into outputs. By convention, outputs of the firm take a positive sign whereas inputs take a negative sign. Using this convention, each firm's production plan can be described by an *n* x 1 column vector, $\mathbf{y}^j\ (j = 1...r)$ which contains both positive and negative entries. The only vectors that the firm may consider are those that are

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[18] A detailed study of labor supply theory is presented in Chapter 16.

feasible given the current state of technology. Sometimes it is convenient to assume each firm produces only one output. But that is not necessary for a more general treatment of production.

Firms are assumed to maximize profits. Production functions are assumed to be sufficiently convex so as to ensure a unique profit maximum for any set of output and input prices. This rules out both increasing returns to scale technologies and constant returns because neither yields a unique maxima. Many general equilibrium models can handle such possibilities, but there is no need to introduce such complexities here. Given these assumptions, the profits for any firm can be written as:

$$\begin{aligned} &\pi_j(\mathbf{p}) = \mathbf{p}\mathbf{y}^j \text{ if } \pi_j(\mathbf{p}) \geq 0 \quad \text{and} \\ &\mathbf{y}^j = \mathbf{0} \text{ if } \pi_j(\mathbf{p}) < 0 \end{aligned} \qquad \textbf{(13.50)}$$

Hence, this model has a "long run" orientation in which firms that lose money (at a particular price configuration) hire no inputs and produce no output. Notice how the convention that outputs have a positive sign and inputs a negative sign makes it possible to phrase profits in a very compact way[19].

Budget constraints and Walras Law

In an exchange model individuals' purchasing power is determined by the values of their initial endowments. Once firms are introduced, we must also consider the income stream that may flow from ownership of these firms. To do so, we adopt the simplifying assumption that each individual owns a predefined share, s_i (where $\sum_{i=1}^{m} s_i = 1$) of the profits of all firms. That is, each person owns an "index fund" that can claim a

[19] As we saw in Chapter 11, profit functions are homogeneous of degree one in all prices. Hence, both output supply functions and input demand functions are homogeneous of degree zero in all prices because they are derivatives of the profit function.

proportionate share of all firms' profits. We can now re-write each individual's budget constraint (from Equation 13.24) as:

$$\mathbf{px}^i = s_i \sum_{j=1}^{r} \mathbf{py}^j + \mathbf{p\bar{x}}^i \quad i = 1...m \tag{13.52}$$

Of course, if all firms were in long-run equilibrium in perfectly competitive industries, all profits would be zero and the budget constraint in Equation 13.52 would revert to that in Equation 13.24. But allowing for long term profits does not greatly complicate our model, so we might as well consider the possibility.

As in the exchange model, the existence of these *m* budget constraints imply a constraint of the prices that are possible – a generalization of Walras Law. Summing the budget constraints in Equation 13.52 over all individuals yields:

$$\mathbf{p}\sum_{i=1}^{m} \mathbf{x}^i(\mathbf{p}) = \mathbf{p}\sum_{j=1}^{r} \mathbf{y}^j(\mathbf{p}) + \mathbf{p}\sum_{i=1}^{m} \mathbf{\bar{x}}^i \tag{13.53}$$

and letting $\mathbf{x}(\mathbf{p}) = \sum \mathbf{x}^i(\mathbf{p}),\ \mathbf{y}(\mathbf{p}) = \sum \mathbf{y}^j(\mathbf{p}),\ \mathbf{\bar{x}} = \sum \mathbf{\bar{x}}^i$ provides a very simple statement of Walras Law:

$$\mathbf{px}(\mathbf{p}) = \mathbf{py}(\mathbf{p}) + \mathbf{p\bar{x}} \tag{13.54}$$

Notice again that Walras Law holds for any set of prices because it is based on individuals' budget constraints.

Walrasian Equilibrium

As before, we define a Walrasian equilibrium price vector $(\mathbf{p}^*)$ as a set of prices at which demand equals supply in all markets simultaneously. In mathematical terms this means that:

$$\mathbf{x}(\mathbf{p}^*) = \mathbf{y}(\mathbf{p}^*) + \mathbf{\bar{x}} \tag{13.55}$$

Initial endowments continue to play an important role in this equilibrium. For example, it is individuals' endowments of potential labor time that provide the most important input for firms' production processes. Determination of equilibrium wage rates is therefore a major output of general equilibrium models operating under Walrasian conditions. Examining changes in wage rates as resulting from changes in exogenous influences is perhaps the most important practical use of such models.

As in the study of an exchange economy, it is possible to use some form of fixed point theorem[20] to show that there exists a set of equilibrium prices that satisfy the *n* equations in 13.55. Because of the constraint of Walras Law, such an equilibrium price vector will be unique only up to a scalar multiple – that is, any absolute price level that preserves relative prices can also achieve equilibrium in all markets. Technically, excess demand functions

$$\mathbf{z}(\mathbf{p}) = \mathbf{x}(\mathbf{p}) - \mathbf{y}(\mathbf{p}) - \bar{\mathbf{x}} \qquad \textbf{(13.56)}$$

are homogeneous of degree zero in prices so any price vector for which $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$ will also have the property that $\mathbf{z}(t\,\mathbf{p}^*) = \mathbf{0}$ for and $t > 0$. Frequently it is convenient to normalize prices so that they sum to one. But many other normalization rules can also be used. In macroeconomic versions of general equilibrium models it is usually the case that the absolute level of prices is determined by monetary factors.

Welfare economics in the Walrasian model with production

Adding production to the model of an exchange economy greatly expands the number of feasible allocations of resources. One way to visualize this is shown in Figure 13.10. There *PP* represents that production possibility frontier for a two-good economy with a

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[20] For some illustrative proofs see K. J. Arrow and F. H. Hahn *General Competitive Analysis* (San Francisco, Holden-Day, 1971) Chapter 5.

fixed endowment of primary factors of production. Any point on this frontier is a feasible. Consider one such allocation, say allocation *A*. If this economy were to produce x_A and y_A we could use these amounts for the dimensions of the Edgeworth exchange box shown inside the frontier. Any point within this box would also be a feasible allocation of the available goods between the two people whose preferences are shown. Clearly a similar argument could be made for any other point on the production possibility frontier.

[Figure 13.10 goes here]

Despite these complications, the first theorem of welfare economics continues to hold in a general equilibrium model with production. At a Walrasian equilibrium there are no further market opportunities (either by producing something else or by reallocating the available goods among individuals) that would make some one individual (or group of individuals) better off without making other individuals worse off. Adam Smith's "invisible hand" continues to exert its logic to ensure that all such mutually beneficial opportunities are exploited (in part because transactions costs are assumed to be zero).

Again, the general social welfare implications of the first theorem of welfare economics are far from clear. There is, of course, a "second theorem" which shows that practically any Walrasian equilibrium can be supported by suitable changes in initial endowments. And one could hypothesize a social welfare function to choose among these. But most such exercises are rather uninformative about actual policy issues.

More interesting is the use of the Walrasian mechanism to judge the hypothetical impact of various tax and transfer policies that seek to achieve specific social welfare criteria. In this case (as we shall see) the fact that Walrasian models stress interconnections among markets, especially among product and input markets, can yield important and often surprising results. In the next section we look at a few of these.

Computable General Equilibrium Models

Two advances have spurred the rapid development of general equilibrium models in recent years. First, the theory of general equilibrium itself has been expanded to include many features of real world markets such as imperfect competition, environmental externalities, and complex tax systems. Models that involve uncertainty and that have a dynamic structure also have been devised, most importantly in the field of macroeconomics. A second, related, trend has been the rapid development of computer power and the associated software for solving general equilibrium models. This has made it possible to study models with virtually any number of goods and types of households. In this section we will briefly explore some conceptual aspects of these models[21]. The Extensions to the chapter describe a few important applications.

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[21] For more detail on the issues discussed here see W. Nicholson and F. Westhoff, "General Equilibrium Models: Improving the Microeconomics Classroom" *Journal of Economic Education*. Summer, 2009: Pages 297-314.