Solutions

Problem Session 5 for Math 29: Independence for RVs, Conditional PDFs, and Smoothing

1. You're going to throw a pizza party for your best friend's birthday. While trying to figure out how many pizzas to order, you ask many of your friends how many pizza slices they ate at the last two pizza parties they were at. Let X be the number of pizza slices eaten at the second to last pizza party, and Y be the number of pizza slices eaten at the more recent (last) pizza party. Suppose you found the joint probability function of X and Y to be as displayed below.

Y/X	1	2	3	4
1	.10	.05	.02	.02
2	.05	.20	.05	.02
3	.02	.05	.20	.04
4	.02	.02	.04	.10

$Y > X$

a. Is this a valid p.f.? Is it a realistic p.f.?
b. What is the probability someone ate more slices at the more recent party than the second to last party?
c. What is the probability someone ate exactly two slices at the second to last party? Exactly two slices at the more recent party?
d. Are X and Y independent? Is your conclusion logical based on how these variables were defined?

a. $\sum_x \sum_y p(x,y) = 1$ & all $p(x,y) \geq 0$ so yes, it is a valid pdf
Realistic pdf? maybe

b. $P(Y > X) = p(1,2) + p(1,3) + p(1,4) + p(2,3) + p(2,4) + p(3,4) = .20$

c. $P(X=2) = .32 \quad P(Y=2) = .32$

d. If $X \perp Y$, $p(x,y) = p_X(x)\, p_Y(y)$. Check using part c.

$P(X=2, Y=2) = .20 \neq .32(.32) = .1024$

Hence X is NOT $\perp$ of Y. People tend to eat similar amts. @ each party so there is more mass than expected along the diagonal where $x = y = k$, $k = 1, \dots, 4$.

2. Suppose Y_1 and Y_2 have joint density given by $f(y_1, y_2) = 2y_1, 0 \le y_1 \le 1, 0 \le y_2 \le 1$, and 0, otherwise.
a. Find the covariance of Y_1 and Y_2.
b. Are Y_1 and Y_2 independent? Provide appropriate justification for your answer.
c. What is the conditional distribution of Y_2 given Y_1?

a. $\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$

$$E(Y_1 Y_2) = \int_0^1 \int_0^1 2y_1^2 y_2 \, dy_2 \, dy_1 = \int_0^1 y_1^2 y_2^2 \Big|_0^1 dy_1 = \int_0^1 y_1^2 \, dy_1 = \frac{1}{3} y_1^3 \Big|_0^1 = \frac{1}{3}$$

$$E(Y_1) = \int_0^1 \int_0^1 2y_1^2 \, dy_2 \, dy_1 = \int_0^1 2y_1^2 y_2 \Big|_0^1 dy_1 = \int_0^1 2y_1^2 \, dy_1 = \frac{2}{3} y_1^3 \Big|_0^1 = \frac{2}{3}$$

$$E(Y_2) = \int_0^1 \int_0^1 2y_1 y_2 \, dy_2 \, dy_1 = \int_0^1 y_1 y_2^2 \Big|_0^1 dy_1 = \int_0^1 y_1 \, dy_1 = \frac{y_1^2}{2} \Big|_0^1 = \frac{1}{2}$$

$$\Rightarrow \text{Cov}(Y_1, Y_2) = \frac{1}{3} - \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{3} - \frac{1}{3} = 0$$

b. To show $\perp$, need to show $f(y_1, y_2) = f_{y_1}(y_1) f_{y_2}(y_2)$.
Cannot use the result from a. to show $\perp$!

$$f(y_1) = \int_0^1 2y_1 \, dy_2 = 2y_1 y_2 \Big|_0^1 = 2y_1, \quad 0 \le y_1 \le 1$$

$$f(y_2) = \int_0^1 2y_1 \, dy_1 = y_1^2 \Big|_0^1 = 1, \quad 0 \le y_2 \le 1$$

So, $f(y_1, y_2) = 2y_1 = 2y_1 \cdot 1 = f(y_1) f(y_2)$, so $Y_1 \perp Y_2$.

c. B/c $Y_1 \perp Y_2$

$$f(y_2 | y_1) = f(y_2) = 1$$

3. Suppose X and Y have the following joint p.d.f.:

$$f(x,y) = 8xy,\ 0 < y < x < 1, \text{and } 0, o.w.$$

a. Find $E(Y|X)$ in general (i.e. you are not given X).
b. Explain how you could use smoothing to find $E(Y)$ using a.
c. Write an integral to solve for $E(Y^2 - 3Y|X)$. Do not solve.

a. $E(Y|X) = \int_y y \cdot f(y|x)\, dy$ Need $f(y|x) = \frac{f(x,y)}{f_x(x)}$. Need $f_x(x)$.

$$f_x(x) = \int_0^x 8xy\, dy = \frac{8xy^2}{2}\Big|_0^x = 4x^3,\ 0 < x < 1$$
$$0,\ o.w.$$

So, $f(y|x) = \frac{8xy}{4x^3} = \frac{2y}{x^2},\ 0 < y < x < 1$, and 0, o.w.

$\Rightarrow$

$$E(Y|X) = \int_0^x y \cdot \frac{2y}{x^2}\, dy = \frac{2}{x^2}\int_0^x y^2\, dy = \frac{2}{3x^2} y^3\Big|_0^x = \frac{2x^3}{3x^2} = \frac{2}{3}x$$
$$0 < x < 1$$

b. By smoothing, we would need to integrate $\frac{2}{3}x$ vs. $f_x(x)$ over all x values

i.e. $E(Y) = E[E(Y|X)] = \int_0^1 \left(\frac{2}{3}x\right) 4x^3\, dx$

c. Just like in part a.

$$E[g(Y)|X] = \int_y g(y)\, f(y|x)\, dy$$
$$= \int_0^x (y^2 - 3y)\left(\frac{2y}{x^2}\right) dy$$

4. When commercial aircraft are inspected, wing cracks are reported as nonexistent, detectable or critical. The history of a particular fleet indicates that 75 percent of planes have no wing cracks, 18 percent have detectable cracks and 7 percent have critical cracks. Five planes from the fleet are randomly selected.

a. What is the probability that one plane has a critical crack, 2 have detectable cracks and 2 have no cracks?

b. What is the covariance between the number of planes with critical cracks and those with no cracks?

c. If the entire fleet is 300 planes, what is the expected value and variance of the number of planes with cracks (detectable or critical)?

$$X \sim \text{Multinomial}\left(5, (\underset{\text{No}}{.75}, \underset{\text{Det}}{.18}, \underset{\text{Crit.}}{.07})\right)$$

a. $P(X_1 = 2, X_2 = 2, X_3 = 1) = \binom{5}{2\ 2\ 1}(.75)^2(.18)^2(.07)^1 = .038272$

b. $\text{Cov}(X_1, X_3) = -np_ip_j = -5(.07)(.75) = -.2625$

c. $n = 300$

$X = X_2 + X_3 \sim \text{Binomial}(300, .25)$

$E(X) = np = 300(.25) = 75$

$V(X) = np(1-p) = 300(.25)(.75) = 56.25$

5. You have just purchased an "assembly required" piece of furniture. It contains 14 pieces which all came off an assembly line on the same day, and which all have equal probability of being defective in some way. You should assume the pieces may be treated independently (if the first piece you check has a defect, that doesn't affect the others in any way, etc.). However, the probability of being defective varies daily, but is known to follow a Beta(1,10) distribution. What is the expected value and variance for the number of defective pieces in a kit like yours?

X = # def. items in set of 14 $\qquad X \sim Bin(14, p)$

p is Beta $(1, 10)$ $\qquad E(p) = \frac{\alpha}{\alpha+\beta} = \frac{1}{11} = .0909$

$$V(p) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{10}{11^2 \cdot 12} = \frac{5}{726} = .006887$$

Use smoothing.

$$E(X) = E\left[E(X|p)\right] = E(14p) = 14E(p) = \frac{14}{11} = 1.2727$$

$$V(X) = E\left[V(X|p)\right] + V\left[E(X|p)\right]$$

$$= E\left[14p(1-p)\right] + V(14p) = 14E(p - p^2) + 14^2 V(p)$$

$$= 14E(p) - 14E(p^2) + 14^2 V(p)$$

$$= 14(.0909) - 14(.01515) + 196(.006887)$$

Note: $E(p^2) = V(p) + (E(p))^2 = .01515$

$$= 1.2726 - .2121 + 1.349852$$

$$= 2.410352$$