Math 12 Spring 2009: Exam 3

Name:

Instructions: There are 4 questions on this exam each of which is scored out of 8 points for a total of 32 points. You may not use any outside materials (eg. notes or books). You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Problem 1. Determine whether or not the sequence converges, and finds its limit if it does converge.

(a) $a_n = \frac{n^2+5}{\sqrt{4n^4+n}}$

(b) $a_n = \frac{n}{(\ln n)^2}$

Proof.

(a) We compute

$$\lim_{n\to\infty} \frac{n^2+5}{\sqrt{4n^4+n}} = \lim_{n\to\infty} \frac{1+5/n^2}{\sqrt{4+1/n^3}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

(b) We compute using L'Hopital's rule twice.

$$\lim_{n\to\infty} \frac{n}{(\ln n)^2} = \lim_{n\to\infty} \frac{n}{2\ln n} = \lim_{n\to\infty} \frac{n}{2} = \infty.$$

so this diverges.

□

Problem 2. Determine whether or not the following series converge absolutely, converge conditionally, or diverge.

(a) $\sum_{n=1}^{\infty} \frac{2n+1}{n^3+n}$

(b) $\sum_{n=1}^{\infty} \frac{5^n}{(\ln 2)^n}$.

Proof.

(a) We apply the limit comparison test to the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{2n+1}{n^3+n} \cdot n^2 = \lim_{n\to\infty} \frac{2n^3+n^2}{n^3+n} = 2.$$

Since this is a finite nonzero value, we know from the limit comparison test that $\sum_{n=1}^{\infty} \frac{2n+1}{n^3+n}$ must also converge.

(b) We apply the root test to see

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{5^n}{(\ln 2)^n}\right|} = \lim_{n\to\infty} \frac{5}{\ln 2} = \frac{5}{\ln 2}.$$

Since $5 > \ln 2$ this quantity is > 1 and by the root test the series diverges.

□

Problem 3. Find the interval and radius of convergence of

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}.$$

Proof. Applying the ratio test to the given series gives

$$\begin{aligned}
\lim_{n\to\infty} \frac{|x+2|^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{|x+2|^n} \\
&= \lim_{n\to\infty} |x+2| \sqrt{\frac{n+1}{n}} \\
&= |x+2| \sqrt{\lim_{n\to\infty} \frac{n+1}{n}} \\
&= |x+2|.
\end{aligned}$$

We have that $R = 1$ and converges for $-3 < x < -1$. We now need to check the endpoints.

Checking $x = -1$ we get the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

which is a p-series with $p = \frac{1}{2}$ so is divergent.

Checking $x = -3$ we get the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

which is an alternating series. The alternating series test tells us that an alternating series converges if and only if

(1) $\lim_{n\to\infty} a_n = 0$

(2) $0 < a_{n+1} \leq a_n$.

Checking (1) we have

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Checking (2) we know that $\sqrt{n+1} > \sqrt{n}$ for all $n > 0$ so we have that

$$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$$

for all $n > 0$.

So we have satisfied both conditions of the alternating series test, so the series is convergent.

Therefore, we have

$$\begin{aligned}
R &= 1 \\
I &= [-3, -1).
\end{aligned}$$

□

Problem 4. Approximate $\int_0^1 \frac{1-e^{-x}}{x} dx$ to within $\frac{1}{100}$.

Proof. We know $e^x = 1 + x + x^2/2 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ so we have

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}.$$

So we have

$$\frac{1-e^{-x}}{x} = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{6} - \cdots\right) = 1 - \frac{x}{2} + \frac{x^2}{6} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(n+1)!}.$$

Integrating term-by-term we have

$$\begin{aligned}\int_0^1 \frac{1-e^{-x}}{x} dx &= \left(x - \frac{x^2}{4} + \frac{x^3}{18} + \cdots\right)_0^1 \\ &= \left(\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n(n!)}\right)_0^1 \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n(n!)}\end{aligned}$$

This is an alternating series, so the remainder is bounded by the next term. Writing out the first few terms we have

$$\int_0^1 \frac{1-e^{-x}}{x} dx = 1 - \frac{1}{4} + \frac{1}{18} - \frac{1}{96} + \frac{1}{600} + \cdots.$$

Since $\frac{1}{600} < \frac{1}{100}$ we have

$$\int_0^1 \frac{1-e^{-x}}{x} dx \approx 1 - \frac{1}{4} + \frac{1}{18} - \frac{1}{96}.$$

□