Math 13, Section 01, Spring 2009

Solutions to the Final Exam

1. **(15 points)** Let S be the plane that contains the points $(0, 0, 1)$, $(4, 2, 0)$, and $(1, -3, 2)$. Find an equation for the line through the point $(1, 0, -2)$ that is perpendicular to S.

Answer. Subtracting the first point from the second and third, we see that the vectors $\vec{a} = \langle 4, 2, -1\rangle$ and $\vec{b} = \langle 1, -3, 1\rangle$ lie in the plane. Their cross product is $\vec{n} = \vec{a} \times \vec{b} = \langle -1, -5, -14\rangle$. Thus, $\vec{n}$ is orthogonal to the plane and hence parallel to the desired line. Replacing $\vec{n}$ by $-\vec{n}$ for convenience, then, the line is given by $\vec{r}(t) = \langle t + 1, 5t, 14t - 2\rangle$.

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2. **(20 points)** Let $f(x, y) = \begin{cases} \dfrac{2x^3 + 3xy - 3y^2}{x^2 + 2y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

(a). Prove that f is **not** continuous at $(0, 0)$.

(b). Compute the directional derivative $D_{\vec{u}}f(0, 0)$, where $\vec{u} = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$.

Answer. (a). Along the y-axis, we have $\lim_{y\to 0} f(0, y) = \lim_{y\to 0} \dfrac{-3y^2}{2y^2} = -\dfrac{3}{2} \neq 0 = f(0, 0)$. Thus, f is discontinuous at $(0, 0)$.

(b). $$D_{\vec{u}}f(0, 0) = \lim_{h\to 0} \frac{f((0, 0) + h\vec{u}) - f(0, 0)}{h} = \lim_{h\to 0} \frac{\dfrac{2h^3}{\sqrt{2}^3} + \dfrac{3h^2}{\sqrt{2}^2} - \dfrac{3h^2}{\sqrt{2}^2}}{h\left(\dfrac{h^2}{\sqrt{2}^2} + \dfrac{2h^2}{\sqrt{2}^2}\right)} = \lim_{h\to 0} \frac{h^3/\sqrt{2}}{h(3h^2/2)} = \frac{\sqrt{2}}{3}.$$

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3. **(25 points)** Find and classify (as local minimum, local maximum, or saddle point) every critical point of the function $f(x, y) = xy^2 - 6x^2 - 3y^2 + 7$.

Answer. We have $f_x = y^2 - 12x$ and $f_y = 2xy - 6y$, both of which are always defined. Setting them both to zero, we get $12x = y^2$ and $(x - 3)y = 0$. By the second equation, either $x = 3$ or $y = 0$. If $x = 3$, then the first equation gives $y^2 = 36$, and hence $y = \pm 6$. If $y = 0$, then the first equation gives $x = 0$. Thus, there are three critical points: $(0, 0)$, and $(3, \pm 6)$.

The second derivatives are $f_{xx} = -12$, $f_{xy} = f_{yx} = 2y$, and $f_{yy} = 2x - 6$. Thus, the discriminant is $D = f_{xx}f_{yy} - f_{xy}^2 = 24(3 - x) - 4y^2$. At $(3, \pm 6)$, we have $D = -4 \cdot 36 < 0$, so that both of those points are saddle points. At $(0, 0)$, we get $D = 3 \cdot 24 > 0$ and $f_{xx} = -12 < 0$. so that there is a local maximum at $(0, 0)$.

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4. **(25 points)** Find the maximum and minimum values of the function $f(x, y) = x^2y$ subject to the constraint $x^2 + y^2 = 9$.

Answer. Let $g(x, y) = x^2 + y^2$ and use Lagrange Multipliers. We have $f_x = 2xy$, $f_y = x^2$, $g_x = 2x$, and $g_y = 2y$. So we have to solve the system of equations $2xy = 2\lambda x$, $x^2 = 2\lambda y$, and $x^2 + y^2 = 9$. The first equation gives $x(y - \lambda) = 0$, so either $x = 0$ or $\lambda = y$. If $x = 0$, then the last equation gives $y^2 = 9$, and so $y = \pm 3$. If $x \neq 0$, then $\lambda = y$, and so the second equation gives $x^2 = 2y^2$, turning the third equation into $3y^2 = 9$. That gives $y = \pm\sqrt{3}$, and hence $x = \pm\sqrt{6}$, with no correlation between the two $\pm$ signs.

Thus, we have six points to check: two of the form $(0, \pm 3)$, and four of the form $(\pm\sqrt{6}, \pm\sqrt{3})$.

We compute $f(0, \pm 3) = 0$, $f(\pm\sqrt{6}, \sqrt{3}) = 6\sqrt{3}$, and $f(\pm\sqrt{6}, -\sqrt{3}) = -6\sqrt{3}$. Thus, the maximum value of f is $6\sqrt{3}$, and the minimum is $-6\sqrt{3}$.

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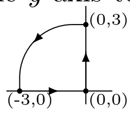
5. **(30 points)** Let C be the path in the xy-plane that begins at $(0, 3)$, runs (counterclockwise) through the second quadrant along the arc of the circle of radius 3 centered at the origin to the point $(-3, 0)$, then moves right along the x-axis to the origin, and finally moves up the y-axis to return to the starting point $(0, 3)$. Compute $\displaystyle\int_C 6x^2y\,dx + (2x^3 - xy)\,dy$.



Answer. Let D denote the quarter-disk bounded by C; note that C is oriented positively with respect to D. Thus, by Green's Theorem, the integral is $\iint_D Q_x - P_y\,dA$, where $P = 6x^2y$ and $Q = 2x^3 - xy$. We compute $Q_x - P_y = 6x^2 - y - 6x^2 = -y$. Thus, converting to polar coordinates, the integral is

$$\iint_D -y\,dA = \int_{\pi/2}^{\pi}\int_0^3 (-r\sin\theta) r\,dr\,d\theta = -\Big(\int_{\pi/2}^{\pi} \sin\theta d\theta\Big)\Big(\int_0^3 r^2\,dr\Big) = \Big(\cos\theta\Big|_{\pi/2}^{\pi}\Big)\Big(\frac{r^3}{3}\Big|_0^3\Big)$$
$$= (-1-0)(9-0) = -9.$$

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6. **(30 points)** Let S denote the sphere in $\mathbb{R}^3$ of radius 2 centered at the origin, oriented outward, and let $\vec{F}(x, y, z) = \langle y^2z, yz^2, x^2e^y\rangle$. Compute $\displaystyle\iint_S \vec{F}\cdot d\vec{S}$.

Answer. We can use the Divergence Theorem, so we compute $\operatorname{div}\vec{F} = 0 + z^2 + 0 = z^2$. Denoting the solid inside the sphere as E and using spherical coordinates, we have

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E z^2\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^2 (\rho\cos\phi)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \Big(\int_0^{2\pi} d\theta\Big)\Big(\int_0^{\pi}\cos^2\phi\sin\phi\,d\phi\Big)\Big(\int_0^2 \rho^4\,d\rho\Big) \qquad [u = \cos\phi, du = -\sin\phi\,d\phi]$$
$$= 2\pi\Big(-\int_1^{-1} u^2\,du\Big)\Big(\frac{\rho^5}{5}\Big|_0^2\Big) = 2\pi\Big(\frac{u^3}{3}\Big|_{-1}^1\Big)\Big(\frac{32}{5} - 0\Big) = 2\pi\cdot\frac{2}{3}\cdot\frac{32}{5} = \frac{128\pi}{15}.$$

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7. **(25 points)** For each of the following vector fields, either find a potential function (i.e., a function that it is the gradient of) or prove that the vector field is not conservative.

(a). $\vec{F}(x, y) = \langle x^2 - \cos(2y), y^3 + 2x\sin(2y)\rangle$.

(b). $\vec{G}(x, y, z) = \langle 2xy - x^2, z^3, 3yz^2\rangle$.

Answer. (a). The 2D curl of $\vec{F}$ is $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 2\sin(2y) - 2\sin(2y) = 0$, on all of $\mathbb{R}^2$. Since $\mathbb{R}^2$ is simply-connected, $\vec{F}$ is conservative, so there is some f with $\nabla f = \vec{F}$.

Then $f_x = x^2 - \cos(2y)$, and so $f(x, y) = \dfrac{x^3}{3} - x\cos(2y) + g(y)$ for some function g. Therefore $f_y = -2x\sin(2y) + g'(y)$, which means $g'(y) = y^3$, and hence we can choose $g(y) = \dfrac{y^4}{4}$. Thus, $f(x, y) = \dfrac{x^3}{3} + \dfrac{y^4}{4} - x\cos(2y)$ is a potential function for $\vec{F}$.

(b). We compute $\operatorname{curl}\vec{G} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \langle 3z^2 - 3z^2, 0 - 0, 0 - 2x\rangle$, which is $\langle 0, 0, -2x\rangle \neq \vec{0}$. Thus, $\vec{G}$ is not conservative.

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8. **(30 points)** Let C be the curve that lies in the surface $z = x^3 - xy + 2$ directly above the boundary of the rectangle $0 \le x \le 2$, $0 \le y \le 1$. Compute $\displaystyle\int_C \sin(x^2)\,dx + yz\,dy - y^2\,dz$, if C is oriented **clockwise** when viewed from above.

Answer. Let S be the portion of the surface $z = x^3 - xy + 2$ bounded by C, so that S is parametrized by $\vec{r}(x, y) = \langle x, y, x^3 - xy + 2\rangle$ for $(x, y) \in [0, 2]\times[0, 1]$. We have $\vec{r}_x = \langle 1, 0, 3x^2 - y\rangle$, and

$\vec r_y = \langle 0, 1, -x\rangle$, so that $\vec r_x \times \vec r_y = \langle y - 3x^2, x, 1\rangle$. However, that vector is pointing upward, which (by the right hand rule) is the wrong way for the orientation on C; so we use $\vec r_y \times \vec r_x = \langle 3x^2 - y, -x, -1\rangle$ instead.

Meanwhile, the vector field $\vec F = \langle \sin(x^2), yz, -y^2\rangle$ has $\operatorname{curl} \vec F = \langle -2y - y, 0, 0\rangle = -3y\vec\imath$. Thus, by Stokes' Theorem,

$$\int_C \vec F \cdot d\vec r = \iint_S (\operatorname{curl} \vec F) \cdot d\vec S = \int_0^2 \int_0^1 -3y(3x^2 - y) + 0 + 0\, dy\, dx = \int_0^2 \int_0^1 3y^2 - 9x^2 y\, dy\, dx$$

$$= \int_0^2 y^3 - \frac{9}{2}x^2y^2\Big|_{y=0}^1 dx = \int_0^2 1 - \frac{9}{2}x^2\, dx = x - \frac{3}{2}x^3\Big|_0^2 = 2 - 12 - (0+0) = -10.$$

BONUS A. (2 points) Let R be region in the first quadrant of the xy-plane bounded to the upper right by $y = 4/x$, to the lower right by $y = x$, to the lower left by $y = 1/x$, and to the upper left by $y = 9x$. Use the transformation $x = \dfrac{\sqrt u}{v}$, $\quad y = v\sqrt u$ to compute $\displaystyle\iint_R \cos(\pi x y)\, dy\, dx$.

Answer. The curve $y = 4/x$ becomes $v\sqrt u = 4v/\sqrt u$, and hence $u = 4$; similarly, $y = 1/x$ becomes $u = 1$. Meanwhile, $y = x$ becomes $v\sqrt u = \sqrt u / v$, and hence $v^2 = 1$, meaning either $v = 1$ or $v = -1$; however, the region R is in the first quadrant and so has $x > 0$, so that it must be $v = 1$. Similarly, $y = 9x$ becomes $v = 3$. Thus, R is the image of the rectangle $S = [1, 4] \times [1, 3]$ in the uv-plane under the given transformation.

The absolute value of the determinant of the Jacobian is $|x_u y_v - x_v y_u| = |(1/(2v\sqrt u)) \cdot \sqrt u - (-\sqrt u / v^2)(v/(2\sqrt u))| = |1/(2v) + 1/(2v)| = 1/v$. Meanwhile, $xy = u$. Thus, the integral is

$$\int_1^4 \int_1^3 \frac{\cos(\pi u)}{v}\, dv\, du = \Big(\ln|v|\Big|_1^3\Big)\Big(\frac{1}{\pi}\sin(\pi u)\Big|_1^4\Big) = (\ln 3 - 0)(0 - 0) = 0.$$

BONUS B. (2 points) Let C be the curve in $\mathbb{R}^3$ that starts at the point $(0, 4, 2)$, goes to the point $(2, 0, 0)$ via $\vec r(t) = \langle t, 4 - t^2, 2 - t\rangle$ for $0 \le t \le 2$; and then goes from $(2, 0, 0)$ to the point $(0, 4, 0)$ along the arc of the parabola $y = 4 - x^2$ in the xy-plane.

Compute $\displaystyle\int_C \cos(\ln x)\, dx + z\, dy + y^2\, dz$.

Answer.

[Note: due to a last-minute change I made, the integral in this problem is actually improper, because $\cos(\ln x)$ is undefined on the plane $x = 0$, which includes two of the endpoints of C. Technically that can be changed by using $(\varepsilon, 4 - \varepsilon^2, 2 - \varepsilon)$ for the starting point, and $(\varepsilon, 4 - \varepsilon^2, 0)$ for the ending point, and then taking the limit as $\varepsilon \searrow 0$. [And the vertical line segment C' that appears below would go from $(\varepsilon, 4 - \varepsilon^2, 0)$ to $(\varepsilon, 4 - \varepsilon^2, 2 - \varepsilon)$.] But, as is common with improper integrals that converge, it all works if you just compute the thing blindly. So, since I did not intend for the problem to be that complicated when I wrote it, I awarded full credit for the simplified method in grading, and I will only do that simplified method here.]

Let C' be the line segment from $(0, 4, 0)$ to $(0, 4, 2)$. So C and C' together form the boundary of a surface S which is sort of a triangular slice of the parabolic cylinder $y = 4 - x^2$. Writing $\vec F = \langle \cos(\ln x), z, y^2\rangle$, Stokes' Theorem tells us that $\displaystyle\int_C \vec F \cdot d\vec r + \int_{C'} \vec F \cdot d\vec r = \iint_S (\operatorname{curl} \vec F) \cdot d\vec S$. We will compute the second and third integrals.

For the second, parametrize C' by $\vec r(t) = \langle 0, 4, t\rangle$ for $t \in [0, 2]$. Then $\vec r\,'(t) = \vec k$, and $y = 4$ on C'; so $\displaystyle\int_{C'} \vec F \cdot d\vec r = \int_0^2 16\, dt = 32$.

Meanwhile, we can parametrize S by $\vec r(x, z) = \langle x, 4 - x^2, z\rangle$, for $0 \le x \le 2$ and $0 \le z \le 2 - x$. We have $\vec r_x = \langle 1, -2x, 0\rangle$ and $\vec r_z = \langle 0, 0, 1\rangle$, and therefore $\vec r_x \times \vec r_z = \langle -2x, -1, 0\rangle$. However, by the right hand rule, this is pointing the wrong way, so instead we use $\vec{r}_z \times \vec{r}_x = \langle 2x, 1, 0\rangle$. We have $\operatorname{curl}\vec{F} = \langle 2y-1, 0, 0\rangle$. Thus,

$$\iint_S (\operatorname{curl}\vec{F})\cdot d\vec{S} = \int_0^2 \int_0^{2-x} (2(4-x^2)-1)(2x)\,dz\,dx = \int_0^2 (14x-4x^3)(2-x)\,dx$$

$$= \int_0^2 28x - 14x^2 - 8x^3 + 4x^4\,dx = 14x^2 - \frac{14}{3}x^3 - 2x^4 + \frac{4}{5}x^5\Big|_0^2 = 8\Big(7 - \frac{14}{3} - 4 + \frac{16}{5}\Big) = \frac{8\cdot 23}{15} = \frac{184}{15}.$$

Finally, then, the original integral is the difference of the two we computed; that is,

$$\int_C \vec{F}\cdot d\vec{r} = \frac{184}{15} - 32 = -\frac{296}{15}.$$

BONUS C. (1 point) Name the largest battle (in terms of numbers of participants and casualties) in the history of human warfare.

Answer. The Battle (or Siege) of Stalingrad, fought between the Nazis and the Soviets from July 1942 through February 1943. Casualty estimates on both sides totalled about 2 million. (By contrast, the Battle of the Bulge, the largest battle involving US forces, had fewer than 200,000 total casualties.)

BONUS D. (1 point) About two weeks ago, a US senator changed political parties. What is that senator's name?

Answer. Arlen Specter. (Senator from Pennsylvania. Was a Republican until recently, but now is a Democrat. Interestingly, he was originally a Democrat long ago but switched to the Republican party back in 1966.)