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Directions: Solve any 3 of the following 4 problems.

Problem #1 For any $A \subset \mathbb{Z}^+$, define $\mathbf{K}A = A$ if A is finite and $\mathbf{K}A = \mathbb{Z}^+$ otherwise. Prove that $\mathbf{K}$ is a closure operator on $\mathbb{Z}^+$.

Problem #1 (Extra credit) Prove that $\mathbf{K}$ does not correspond to a metric.

Problem #2 Let $X = \mathbb{R}$ with the topology corresponding to the following closure operator: $\mathbf{K}_X A = A \cup \{0\}$ if A is nonempty and $\emptyset$ otherwise. Similarly, let $Y = \mathbb{R}$ with the topology from the following closure operator: $\mathbf{K}_Y A = A \cup \{1\}$ if A is nonempty and $\emptyset$ otherwise. You may assume that $\mathbf{K}_X$ and $\mathbf{K}_Y$ are closure operators on X and Y, respectively.

(a) Prove $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ are both connected.

(b) Is the function $f : X \to Y$ defined by $f(x) = x$ continuous (with respect to the topologies above)? Prove your answer.

(c) Give a continuous function $g : X \to Y$ that agrees with f at all but one point. Prove your g is continuous, but not onto or 1-1.

(d) Prove there is a continuous function $h : X \to Y$ that is onto and 1-1.

Problem #3 Let X be any set with the discrete topology.

(a) Let Y be any set, and let f be any function from X to Y. Prove f is continuous.

(b) For any $p \in X$, define $g_p : X \to X$ by $g_p(x) = p$ for all $x \in X$. Prove g_p is a contraction.

(c) Prove every contraction on X is of the form g_p for some $p \in X$.

(d) Conclude that any contraction of a discrete space has a unique fixed point.

Problem #4 Given $A \subset \mathbb{R}$, let $f(x) = (x + \sqrt{2})/2$ for all $x \in A$.

(a) Is f a contraction for any of the following sets? $A = \mathbb{Q}, A = \mathbb{R} - \mathbb{Q}, A = \mathbb{R}$.

(b) If you said f was a contraction for any of the sets in part (a), prove your assertion(s).

(c) Find all fixed points for the set(s) in part (b). Prove you have provided all of them.