For each question, I give a model solution (so you can see the level of detail that I expect from you in the exam) and some comments.

1. Find the equation of the plane that passes through the point \( (1, 0, 2) \) and contains the line whose vector equation is
\[
r(t) = \langle 1 - t, t + 2, -3t \rangle.
\]

Solution: One vector parallel to the plane we want is \( \langle -1, 1, -3 \rangle \) (the vector in the direction of the given line). Another point in the plane is
\[
r(0) = \langle 1, 2, 0 \rangle
\]
which tells us that a second vector parallel to the plane is the vector from \( (1, 0, 2) \) to \( (1, 2, 0) \), namely \( \langle 0, 2, -2 \rangle \).

Therefore a vector perpendicular to the plane is
\[
\mathbf{n} = (-1, 1, -3) \times \langle 0, 2, -2 \rangle
\]
\[
= ((1)(-2) - (-3)(2), (-3)(0) - (-1)(-2), (-1)(2) - (1)(0))
\]
\[
= \langle 4, -2, -2 \rangle.
\]
The equation of the plane is therefore
\[
4(x - 1) - 2(y - 0) - 2(z - 2) = 0
\]
or
\[
4x - 2y - 2z = 0.
\]

Comments:
If you found a different vector \( \mathbf{n} \) (that is, some multiple of the above \( \mathbf{n} \) such as \( \langle 2, -1, -1 \rangle \)), then you will get a different equation of the plane, such as
\[
2x - y - z = 0.
\]
The one you get should be some constant multiple of the equation above.

There are other ways to find a vector \( \mathbf{n} \) that is perpendicular to the plane. A reasonable way is to find two points on the given line, such as
\[
r(0) = \langle 1, 2, 0 \rangle \quad \text{and} \quad r(1) = \langle 0, 3, -3 \rangle.
\]
Then, together with the given point \( (1, 0, 2) \) we now have three points in the plane. If we call these \( A, B \) and \( C \), then the vector
\[
\vec{AB} \times \vec{AC}
\]
is perpendicular to the plane. (This was how we found the equation on a plane in the first example from class.)
2. (a) Find the point of intersection of the curves

\[ r(t) = (1 - t, t + 2, -3t) \quad \text{and} \quad s(u) = (u, 2u, u^2 - u). \]

(b) Find the acute angle between the tangent lines to these two curves at the point of intersection. (You may give your answer using inverse trigonometric functions if necessary.)

Solution: (a) We set \( r(t) = s(u) \), that is

\[
\begin{align*}
1 - t &= u \\
t + 2 &= 2u \\
-3t &= u^2 - u
\end{align*}
\]

Substituting the first into the second, we get

\[
t + 2 = 2(1 - t)
\]

so

\[3t = 0\]

and so \( t = 0 \). Therefore \( u = 1 \). (Check: it is then true that \(-3t = u^2 - u\).)

The point of intersection is therefore \( r(0) = (1, 2, 0) \).

(b) The tangent vectors are

\[ r'(t) = (-1, 1, -3) \]

so

\[ r'(0) = (-1, 1, -3) \]

and

\[ s'(u) = (1, 2, 2u - 1) \]

so

\[ s'(1) = (1, 2, 1). \]

Then

\[
\cos(\theta) = \frac{(-1, 1, -3) \cdot (1, 2, 1)}{|(-1, 1, -3)||1, 2, 1|} = \frac{-2}{\sqrt{11}\sqrt{6}}
\]

Therefore the angle between the vectors is

\[ \theta = \cos^{-1}\left(\frac{-2}{\sqrt{66}}\right). \]

This is not an acute angle so the angle we want is actually

\[ \pi - \theta = \cos^{-1}\left(\frac{2}{\sqrt{66}}\right). \]
Comments: In part (a), to find the point of intersection between the curves (or indeed to figure out if the curves intersect at all) we have to allow for different inputs to the two functions since we don’t care that the two particles represented by these functions are at the intersection point at the same time. If there turns out to be no solution to the three equations (that is, there are no values for \( t \) and \( u \) that make all three true) then the curves would not intersect at all. (In our example we do the ‘Check’ to ensure that our solution \((t = 0, u = 1)\) really does satisfy all three equations.)

In part (b), a common mistake is to forget to substitute in \( u = 1 \) to get \( s'(1) \). This is important because we are interested in the tangent vectors at the point \((1,2,0)\). The formula \( s'(u) \) represents the tangent vector at all points along the curve (depending on \( u \)) so we need to pick out the one we want which is \( s'(1) \). We also have to do this to \( r'(t) \) to get \( r'(0) \), but in this case, \( r'(t) \) is a constant vector so there is no change. But, your answer to this question should definitely not depend on \( u \)!

The last part about finding the acute angle is there because there are always two different angles between two lines (the angles sum to 180 degrees). The question specified that we wanted the acute angle (i.e. the one less than 90 degrees). The inverse cosine of a negative number is larger than 90 degrees, so we want the other one. We can get this by taking inverse cosine of the corresponding positive amount because

\[
\cos(\pi - \theta) = -\cos(\theta). 
\]

3. Let \( r(t) \) be a vector function that parameterizes a curve \( C \). Give the definition of the unit tangent vector \( T(t) \) and prove that \( T(t) \) is always orthogonal to its derivative \( T'(t) \).

Solution: The unit tangent vector is given by

\[
T(t) = \frac{r'(t)}{|r'(t)|}.
\]

Since \( T(t) \) is a unit vector for all \( t \), we have

\[
T(t) \cdot T(t) = 1.
\]

Differentiating with respect to \( t \):

\[
2T'(t) \cdot T(t) = 0
\]

and so \( T'(t) \) is perpendicular to \( T(t) \).

Comments: See Example 4 on page 874 for essentially the same problem. There was also a homework problem on this topic. This approach is much better than trying to differentiate \( T(t) \) directly, though that is possible. If you were to do this that way, you want to be as economical with the notation as possible. Let \( r = \langle f, g, h \rangle \) (omitting the dependence on \( t \) to make things more readable). Then

\[
T = \frac{1}{\sqrt{f'^2 + g'^2 + h'^2}} \langle f', g', h' \rangle.
\]
Therefore, by the Product Rule:

\[ T' = \frac{-(f' f'' + g' g'' + h' h'')}{(f'^2 + g'^2 + h'^2)^{3/2}} (f', g', h') + \frac{1}{\sqrt{f'^2 + g'^2 + h'^2}} (f'', g'', h''). \]

To figure out if \( T \) and \( T' \) are perpendicular, we take the dot product

\[ T \cdot T' = \frac{-(f' f'' + g' g'' + h' h'')}{(f'^2 + g'^2 + h'^2)^{2}} (f'^2 + g'^2 + h'^2) + \frac{1}{(f'^2 + g'^2 + h'^2)} (f' f'' + g' g'' + h' h'') = 0. \]

So they are perpendicular as claimed. But the other way is a lot easier!

4. **Find an arc length parametrization of the helix**

\[ r(t) = \langle 2t, \cos(\pi t), \sin(\pi t) \rangle. \]

**Solution:** We have

\[ r'(t) = \langle 2, -\pi \sin(\pi t), \pi \cos(\pi t) \rangle \]

and

\[ |r'(t)| = \sqrt{4 + \pi^2 \sin^2(\pi t) + \pi^2 \cos^2(\pi t)} = \sqrt{4 + \pi^2}. \]

Therefore

\[ s(t) = \int_{u=0}^{u=t} |r'(u)| \, du = \int_{u=0}^{u=t} \sqrt{4 + \pi^2} \, du = \left[ \sqrt{4 + \pi^2} u \right]_{u=0}^{u=t} = \sqrt{4 + \pi^2} t. \]

So

\[ t = \frac{s}{\sqrt{4 + \pi^2}}. \]

So the arc length parametrization is

\[ u(s) = r(t(s)) = \left\langle \frac{2s}{\sqrt{4 + \pi^2}}, \cos\left(\frac{\pi s}{\sqrt{4 + \pi^2}}\right), \sin\left(\frac{\pi s}{\sqrt{4 + \pi^2}}\right) \right\rangle. \]

**Comments:** This is a standard application of the procedure for finding the arc length parametrization. The most confusing part is understanding the limits of the integral that defines \( s(t) \). If it is not specified, you can pick any choice of starting point for the integration. This corresponds to choosing the \( t \)-value for which the arc length will be 0. Here I took that \( t \)-value to be 0. This means that the arc length is measured along the curve starting from \( r(0) = (0, 1, 0) \).

In the case that the starting point is not specified, you could also use an indefinite integral to define \( s(t) \) (and you can choose any value of the constant of integration you like). So we could have just said

\[ s(t) = \int \sqrt{4 + \pi^2} \, dt = \sqrt{4 + \pi^2} t. \]

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5. Find the curvature of the parabola \( y = ax^2 + bx + c \) at its apex \( x = \frac{-b}{2a} \).

Solution: We have

\[ \kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}. \]

Then

\[ f'(x) = 2ax + b, \quad f''(x) = 2a \]

so

\[ \kappa(x) = \frac{|2a|}{(1 + (2ax + b)^2)^{3/2}}. \]

Setting \( x = \frac{-b}{2a} \), we get

\[ \kappa(-b/2a) = |2a|. \]

Comments: It’s easiest to use this general formula for finding the curvature of a curve given by \( y = f(x) \) (that is, the graph of \( f \)). If you didn’t remember that formula, you can still figure out the curvature using a parametrization of the parabola, such as

\[ \mathbf{r}(t) = \langle t, at^2 + bt + c \rangle. \]

Then

\[ \mathbf{r}'(t) = \langle 1, 2at + b \rangle \]

and

\[ \mathbf{r}''(t) = \langle 0, 2a \rangle. \]

Now we can apply the formula

\[ \kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}. \]

This looks weird because \( \mathbf{r}'(t) \) and \( \mathbf{r}''(t) \) are vectors in \( \mathbb{R}^2 \) not in \( \mathbb{R}^3 \). The way we take the cross product then is to treat these as vectors in \( \mathbb{R}^3 \) with third component equal to zero. Therefore

\[ \mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 1, 2at + b, 0 \rangle \times \langle 0, 2a, 0 \rangle \]

\[ = \langle 0, 0, 2a \rangle \]

This gives us

\[ \kappa(t) = \frac{|2a|}{(1 + (2at + b)^2)^{3/2}} \]

(which should look familiar from above) and so

\[ \kappa(-b/2a) = |2a|. \]

(Notice that the \( t \)-value for the apex is \( -b/2a \) since \( x = t \)).