Math 28 Spring 2008: Exam 1

Instructions: Each problem is scored out of 10 points for a total of 50 points. You may not use any outside materials(eg. notes or calculators). You have 50 minutes to complete this exam.

Problem 1. Let $A \subseteq \mathbb{R}$ be bounded above. Show that $\sup(A) \in \overline{A}$.

Proof. Let A be bounded above and let $s = \sup(A)$ which exists by the completeness of $\mathbb{R}$. Then we need to see that $s \in \overline{A} = A \cup L$ where L is the set of limit points of A. Since s can either be in A or not in A, we examine each case separately.

If $s \in A$ then clearly $s \in \overline{A}$.

If $s \notin A$, then we will show that s is a limit point. Let $\epsilon > 0$, then by the characterization of supremum, there exists an $a \in A$ such that $s - \epsilon < a$. Since $s \notin A$ we have $a \neq s$ and hence $V_\epsilon(s)$ intersects A in a point not s. Therefore, s is a limit point of A and hence $s \in \overline{A}$. □

Problem 2. Let $(a_n) \to a$ and $(b_n) \to b$ be convergent sequences. Show directly that $(a_n - b_n) \to a - b$. (i.e. without using the Algebraic Limit theorem).

Proof. We know that $(b_n) \to b$ and hence for all $\epsilon > 0$ $\exists N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have $|b_n - b| < \frac{\epsilon}{2}$. So we also have that for all $n \geq N_1$ we have

$$|-b_n - (-b)| = |b - b_n| = |b_n - b| < \frac{\epsilon}{2}.$$

Similarly, since $(a_n) \to a$ (and the same $\epsilon > 0$) there exists $N_2 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\epsilon}{2} \quad \forall n \geq N_2$$

Let $N = \max(N_1, N_2)$. Then for all $n \geq N$ we have

$$\begin{aligned} |(a_n - b_n) - (a - b)| &= |(a_n - a) - (b_n - b)| \\ &= |(a_n - a) + (b - b_n)| \\ &\leq |a_n - a| + |b - b_n| \quad \text{(Triangle Inequality)} \\ &= |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Problem 3.

(a) State the definition of a compact set and the characterization of compact sets by the Heine-Borel Theorem.

(b) Show that the union of finitely many compact sets is compact.

Proof.

(a) Compact sets are sets such that every sequence of elements contained in the set has a subsequence which converges to a limit which is in the set. The Heine-Borel Theorem states that a set is compact if and only if it is closed and bounded.

(b) Let $K_i \subset \mathbb{R}$ for $1 \le i \le n$ be compact sets where $n \in \mathbb{N}$. Since a finite union of closed sets is closed, we have that $K = \cup_{i=1}^{n} K_i$ is closed. Since K_i is compact for $1 \le i \le n$ it is bounded, so there exists $M_i \in \mathbb{R}$ such that $|a| \le M_i$ for all $a \in K_i$ for $1 \le i \le n$. Let $M = \max(M_1, \ldots, M_n)$. The maximum of a finite set exists, so M exists. Consider $a \in K$. Then $a \in K_i$ for some i and hence $|a| \le M_i \le M$. So K is bounded. Using the Henie-Borel theorem K is compact since it is closed and bounded.

Or using open covers: a subset of $\mathbb{R}$ is compact if and only if every open cover has a finite subcover. Let $\{O_j\}$ be an open cover of $K = \cup_{i=1}^{n} K_i$. Then there exists a finite collection $\{O_{k_i}\}_{k_i=1}^{a_i}$ for each K_i. Hence the collection $\cup_{i=1}^{n}\{O_{k_i}\}_{k_i=1}^{a_i}$ covers all of the K_i and hence covers K. Since the finite union of a finite number of objects is finite. This is a finite subcover. So, by the characterization of compact, K is compact.

□

Problem 4. Prove the $\sum_{n=1}^{\infty} a_n$ is convergent if and only if for any $m \in \mathbb{N}$ $\sum_{n=1}^{\infty} a_{m+n}$ is convergent. Moreover, $\sum_{n=1}^{\infty} a_n$ converges to $a_1 + \cdots + a_m + \sum_{n=1}^{\infty} a_{n+m}$.

Proof. Let $m \in \mathbb{N}$. Let S_n be the nth partial sum of $\sum_{n=1}^{\infty} a_n$ and let T_n be the nth partial sum of $\sum n = 1^{\infty} a_{m+n}$.

First showing $(\Rightarrow)$: $\sum a_n$ converges if and only if (S_n) is Cauchy. So we have for all $n, k \ge N \in \mathbb{N}$ we have

$$|S_n - S_k| = |a_{k+1} + \cdots + a_n| < \epsilon$$

Note also that $n+m, k+m$ satisfy $n+m, k+m \ge N$ and hence

$$|T_n - T_k| = |a_{k+m+1} + \cdots + a_{n+m}| < \epsilon.$$

Therefore (T_n) is Cauchy and hence $\sum_{n=1}^{\infty} a_{m+n}$ converges.

$(\Leftarrow)$: Conversely, if $n+m, k+m \ge N$ implies

$$|T_n - T_k| = |a_{k+m+1} + \cdots + a_{n+m}| < \epsilon$$

then $n, k \ge N - m$ implies

$$|S_n - S_k| = |a_{k+1} + \cdots + a_n| < \epsilon$$

and hence (S_n) is Cauchy. Note that we know $N > m$ since $n, k \in \mathbb{N}$.

For the value of convergence we simply look at the terms of the sequence.

$$\sum_{n=1}^{\infty} a_{n+m} = a_1 + \cdots + a_m + \sum_{n=1}^{\infty} a_n.$$

So if $\sum_{n=1}^{\infty} a_{n+m} = A$ and $\sum_{n=1}^{\infty} a_n = B$, then we have

$$A = a_1 + \cdots + a_m + B.$$

□

Problem 5.

(a) State the Monotone Converge Theorem.

(b) Define $a_1 = 1$ and $a_n = 3 - \frac{1}{a_{n-1}}$ for $n \geq 2$. Determine the convergence or divergence of the sequence (a_n).

Proof.

(a) The Monotone Convergence theorem says that every bounded monotonic sequence is convergent.

(b) We will show the sequence is increasing and bounded.

For increasing we proceed by induction. We have $a_1 = 1$ and $a_2 = 2$ and hence the base case $a_2 > a_1$. So assume $a_n > a_{n-1}$ and hence $\frac{1}{a_{n-1}} > \frac{1}{a_n}$ for all $n \leq N$. Then consider

$$a_{n+1} = 3 - \frac{1}{a_n} > 3 - \frac{1}{a_{n-1}} = a_n.$$

So the sequence is increasing and hence monotone.

For boundedness, we know that $a_1 = 1$ and (a_n) is increasing and will show that $1 \leq a_n \leq 3$ for all $n \in \mathbb{N}$. Consider

$$a_{n+1} = 3 - \frac{1}{a_n}$$

Since a_n in increasing and at least 1, the upper bound of 3 holds since we are always subtracting a positive quantity from 3. Since $a_1 = 1$ and (a_n) is increasing, the lower bound holds.

□