Some Basic Concepts about Functions

Please read this handout carefully and ask questions in office hours about *any* aspect you don't fully understand.

§1: Basic Definitions

Definition. Let A and B be sets.

- A ***function*** $f : A \to B$ means that for every $a \in A$, there is a unique $f(a) \in B$. We call A the ***domain*** of f.
- Two functions $f, g : A \to B$ are ***equal***, written $f = g$, if $f(a) = g(a)$ for all $a \in A$.

Definition. Given functions $f : A \to B$ and $g : B \to C$, the ***composition*** $g \circ f : A \to C$ is the function defined by

$$(g \circ f)(a) = g(f(a)) \quad \text{for all } a \in A.$$

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$ be $g(x) = \sin(x)$. Then $(g \circ f)(x) = \sin(x^2)$ and $(f \circ g)(x) = \sin(x)^2 = \sin^2(x)$.

§2: One-to-One and Onto

Definition. A function $f : A \to B$ is:

- ***one-to-one*** or ***1-1*** or ***injective*** if for all elements $a, b \in A$,
$$a \neq b \Longrightarrow f(a) \neq f(b).$$
- ***onto*** or ***surjective*** if for every $b \in B$ there is at least one $a \in A$ such that $f(a) = b$.

Facts about One-to-One.

- One-to-one is equivalent to saying that $f : A \to B$ is one-to-one if for all elements $a, b \in A$,
$$f(a) = f(b) \Longrightarrow a = b.$$

Example. $\sin : \mathbb{R} \to \mathbb{R}$ is not one-to-one but $\sin : [-\pi/2, \pi/2] \to \mathbb{R}$ is.

Facts about Onto.

- The ***range*** of a function $f : A \to B$ is the set
$$R(f) = \{f(a) \mid a \in A\} \subseteq B.$$
- A function $f : A \to B$ is onto if and only if $R(f) = B$, i.e., its range is as big as possible. Be sure you understand this.

Example. $\sin : \mathbb{R} \to \mathbb{R}$ is not onto but $\sin : \mathbb{R} \to [-1, 1]$ is onto since $[-1, 1]$ is the range of sin. Be sure you understand this.

§3: Inverse Functions and Bijections

Definition.

- Given any set A, the ***identity function*** $I_A : A \to A$ is the function defined by $I_A(a) = a$ for every $a \in A$.
- Functions $f : A \to B$ and $g : B \to A$ are called ***inverse functions*** if $g \circ f = I_A$ and $f \circ g = I_B$.
- A function is ***bijective*** if it is one-to-one and onto.

Facts about Inverse Functions.

- Functions $f : A \to B$ and $g : B \to A$ are inverse functions if and only if
$$\begin{aligned} g(f(a)) &= a \quad \text{for all } a \in A \\ f(g(b)) &= b \quad \text{for all } b \in B. \end{aligned}$$
- If $f : A \to B$ has an inverse function, then the inverse is unique and is denoted $f^{-1} : B \to A$.
- If $f : A \to B$ has an inverse function f^{-1}, then f is the inverse of f^{-1}, i.e., $(f^{-1})^{-1} = f$.

Examples of Inverse Functions.

- Let $\mathbb{R}_{>0} = (0, \infty)$ be the set of positive real numbers. Then the functions $e^x = \exp(x)$ and $\ln(x)$ from calculus give inverse functions
$$\exp : \mathbb{R} \to \mathbb{R}_{>0}, \quad \ln : \mathbb{R}_{>0} \to \mathbb{R}$$
since
$$\begin{aligned} \exp(\ln(x)) &= e^{\ln(x)} = x \quad \text{for all } x \in \mathbb{R}_{>0} \\ \ln(\exp(x)) &= \ln(e^x) = x \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$
- $\sin : [-\pi/2, \pi/2] \to [-1, 1]$ and $\sin^{-1} = \arcsin : [-1, 1] \to [-\pi/2, \pi/2]$ are inverse functions. You studied these functions in calculus.

Facts about Bijections and Inverse Functions

- $f : A \to B$ has an inverse function if and only f is bijective.
- If $f : A \to B$ and $h : B \to C$ are bijections, then the composition $h \circ f : A \to C$ is a bijection. Furthermore, its inverse function is given by $(h \circ f)^{-1} = f^{-1} \circ h^{-1}$.

Example of a Bijection. The squaring function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^2$ is bijective (do you see why?). Hence it has an inverse function $f^{-1} : [0, \infty) \to [0, \infty)$, which is clearly the square root function, i.e., $f^{-1}(x) = \sqrt{x}$.

The "Facts" stated above are theorems whose proofs we will skip. Math 220 covers this material in detail. See also page 156 of the linear algebra text.