Connect Hubble expansion with Galaxy counts

(1)

(i) we have a conservation law

$\rho R^3 = \text{const}$

if we ÷ by the mass of a galaxy this reads $n R^3 = \text{const}$

or $n = \alpha R^{-3}$ [α is the const]

(ii) then $dn/dt = -3\alpha R^{-4}\, dR/dt$

$$\left|\frac{\dot{n}}{n}\right| = + \frac{3\alpha R^{-4}\dot{R}}{\alpha R^{-3}}.$$

$$|\dot{n}/n| = +3\, \dot{R}/R = +3H$$

[H is the hubble const]

let's write H as $1/t_H$.

[the Hubble age].

Then, from previous work in Problem set 1, $|n/\dot{n}| = 3.4$ billion years $= t_H/3$

$t_H = 3 \times 3.4 = 10.2$ billion years

"Hubble's law cubed" is NOT universal

Suppose we observe 6 galaxies: Galaxy A, B, C, D, E and F. We measure their distances D (in megaparsecs) and velocities from us V (in kilometers / second). Here's what we find

HUBBLE'S COSMIC TABLE

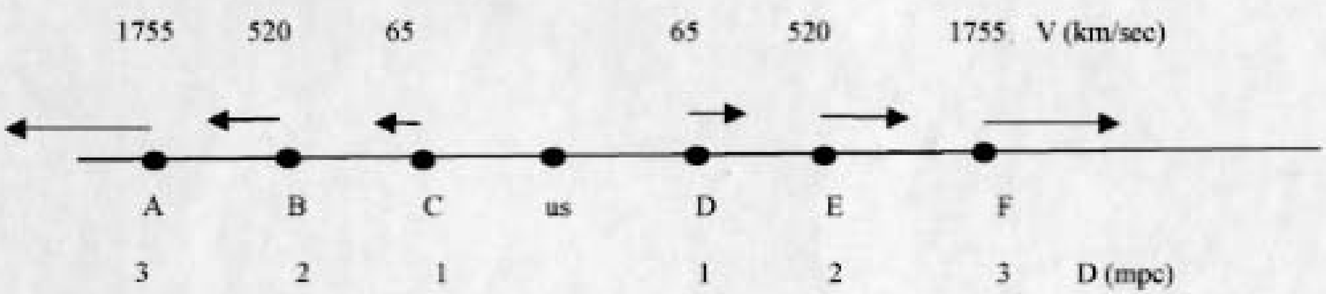
Galaxy	D	V
A	3	1755
B	2	520
C	1	65
D	1	65
E	2	520
F	3	1755

Now we do a little math: we divide V by D^3 and we get 65 in every case. So we conclude

$V = H\,D^3$

Where H = 65. This, of course, is not Hubble's law.

Here's a diagram illustrating what we have found:



Now we want to ask a new question: <u>what would an alien cosmologist living on another galaxy find</u>? To be specific, let's suppose that our alien's name is Elbbuh, and it (they don't have sexes) lives in galaxy D. We need to find

(a) the *distances Elbbuh measures*. These are the distances of each galaxy from D. They are

Galaxy	Distance from galaxy D (mpc)
A	4
B	3
C	2
Us	1
E	1
F	2

(b) the *velocities Elbbuh measures*. These are the velocities of all the galaxies relative to D.

These are a little harder to find. Let's start with "us" -- this is pretty easy. Since D is going 65 km/sec away from us, we are going 65 km/sec away from it.

Now let's study galaxy C. In our above diagram, it is going 65 km/sec to the left, and D is going 65 km/sec to the right. So the velocity of C relative to D is 65+65=130 km/sec.

Similarly, in the above diagram, E is going 520 km/sec to the right, and D is going 65 km/sec to the right. D is "trying to keep up with E but not going fast enough to do so." So the velocity of E relative to D is 520 – 65 = 455 km/sec.

We keep on in the same way, and finally reach the following results

ELBBUH'S COSMIC TABLE

Galaxy	Distance from galaxy D (mpc)	Velocity relative to galaxy D (km/sec)	Velocity divided by D cubed
A	4	1820	28.4
B	3	585	21.7
C	2	130	16.3
us	1	65	65
E	1	455	455
F	2	1690	211

Now here's the final step. Let's divide the velocity by the distance cubed to find "Elbbuh's law." The results are shown in the right-hand column of the above table. Notice that one of the entries is 65 – but the others are not. This proves that

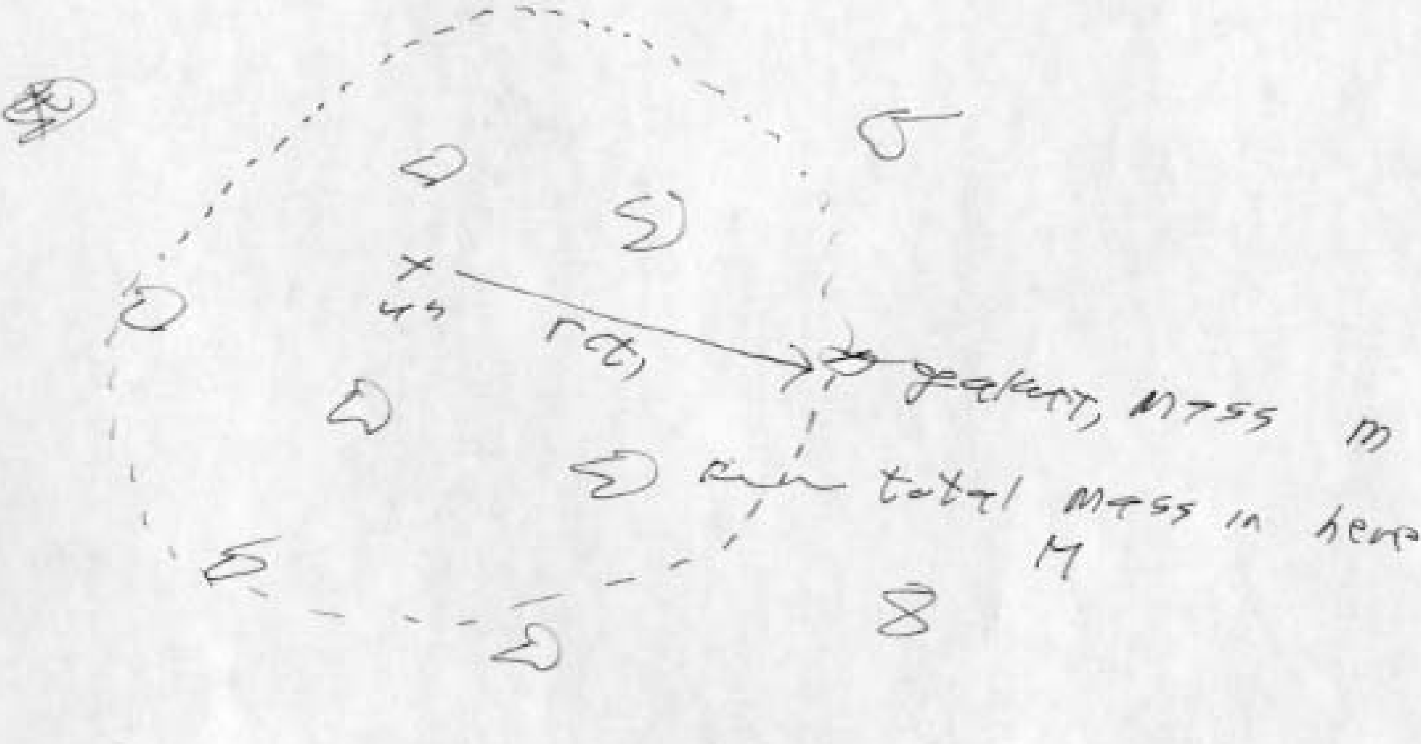
$\text{Velocity}_{\text{relative to D}}$ does NOT equal $(65)(\text{Distance}_{\text{relative to D}})^3$

So Elbbuh's law is not the same as Hubble's law. This proves that '*Hubble's law cubed*' *is not universal.*

(3) Friedman eqs in the early universe

(a)

we procede just like in class: find the force of gravity on one galaxy:



The force is $F = -\frac{GMm}{r^2}$

because the force decelerates the galaxy's motion

$M = \frac{4}{3}\pi r^3 \rho$ so $F = -\frac{4}{3}\pi G m \rho r$

and $F = ma = m\ddot{r}$ so

$\ddot{r} = -\frac{4}{3}\pi G \rho r$

in terms of the expansion parameter:

$r(t) = R(t)\, r(t_0)$

$\ddot{r}(t) = \ddot{R}(t)\, r(t_0)$

so we get

$\ddot{R}(t) = -\frac{4}{3}\pi G \rho(t)\, R(t)$

This is exactly what we did in class. Here's the new twist: write

$\rho(t) = \dfrac{\text{const}}{R^4(t)}$ where the

constant is $\rho(t_0)\, \underbrace{R^4(t_0)}_{=1} = \rho(t_0)$

so we get

$$\boxed{\ddot{R}(t) = -\frac{4}{3}\pi G \rho(t_0) / R^3(t)}$$ new 1st eq

multiply by $\dot{R}$

$$\underbrace{\ddot{R}\,\dot{R}}_{A} = -\frac{4}{3}\pi G \rho(t_0)\, \underbrace{\frac{\dot{R}}{R^3}}_{B}$$

A is the derivative of $\frac{1}{2}\dot{R}^2$

B is the derivative of $\left[-\frac{1}{2}R^{-2}\right]$

so the above equation can be written

$$\frac{d}{dt}\left[\frac{1}{2}\dot{R}^2\right] = \frac{d}{dt}\left[-\frac{4}{3}\pi G\rho(t_0)\right]\left[-\frac{1}{2}R^{-2}\right]$$

so

$$\frac{1}{2}\dot{R}^2 = \left[-\frac{4}{3}\pi G\rho(t_0)\right]\left[-\frac{1}{2}R^{-2}\right] + \text{const}$$

$$\boxed{\dot{R}^2 = \frac{4}{3}\pi G\rho(t_0)\frac{1}{R^2(t)} + \text{const}}$$

2nd new ~~...~~ Friedman eq

(b) Let's verify that $R(t) = \alpha t^{1/2}$ solves the equation if "const" is zero.

$$\frac{dR}{dt} = \frac{d}{dt}\left[\alpha t^{1/2}\right] = \frac{1}{2}\alpha t^{-1/2}$$

so $\dot{R}^2 = \frac{1}{4}\alpha^2 t^{-1}$ ← {is the left-hand side the right hand side is

$$\frac{\frac{4}{3}\pi G\rho(t_0)}{\left[\alpha t^{\frac{1}{2}}\right]^2} = \frac{4}{3}\pi G\rho(t_0) \Big/ \alpha^2 t$$

equating

$$\frac{1}{4}\alpha^2 t^{-1} = \frac{4}{3}\pi G\rho(t_0)\frac{1}{\alpha^2 t}$$

the t's cancel, and if

$$\alpha^4 = \frac{16}{3}\pi G\rho(t_0)$$

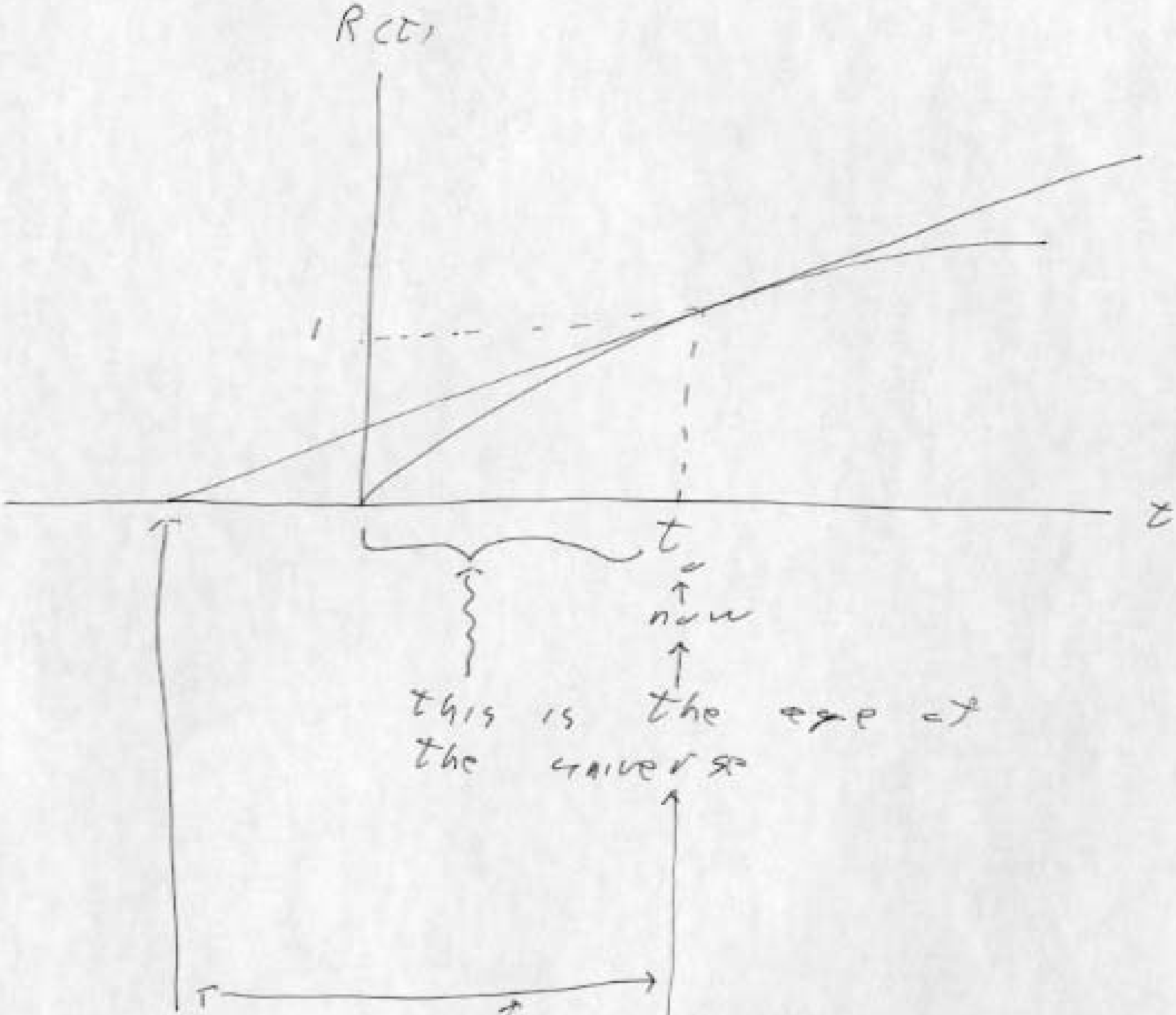
the two sides are equal. Thus

$$\alpha = \left[\frac{16}{3}\pi G\rho(t_0)\right]^{\frac{1}{4}}$$

c) yes, there was a big bang, since when t=0 R was zero and the density was infinite

$\boxed{4}$ The Hubble Age

I can think of 2 ways to do this. Either is fine



textbook says this is $T_H = \frac{1}{H} = \frac{R}{\dot{R}}$

but why?

one way: intuitively

the ~~red~~ tangent line has constant slope, and

$$\underbrace{r(t)}_{\text{dist to a galaxy}} = r(t_0)\, \underbrace{R(t)}_{\text{dist now}}$$

so $\frac{d}{dt} r(t) = v(t) = r(t_0)\dot{R}$

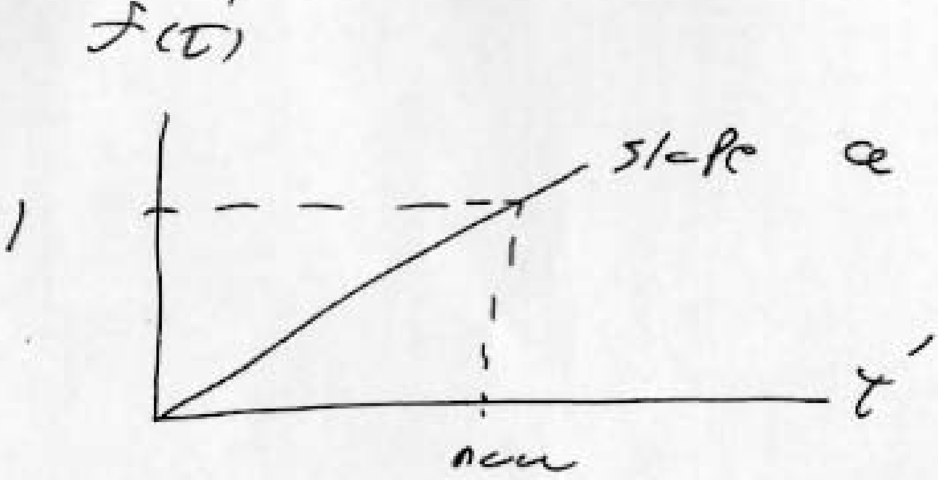
if $\dot{R}$ is constant, $v(t)$ is too

and we proved in class that, if galaxies move with constant velocity, the age of the universe

is $T_H = \frac{1}{H}$

second way: mathematically

let's first draw a graph of the straight lines against a different time axis t'



the eq for the line is $f(t') = at'$

and it attains the value 1 at

$$t' = \frac{1}{a} = \frac{1}{\text{slope}}$$

So, on the graph on the previous page, the interval of time which the textbook calls T_H is $\frac{1}{\text{slope}}$

$= \frac{1}{\dot{R}}$ But we can multiply this by $R(t_0)$ since this $= 1$, so we get

$T_H = \frac{R}{\dot{R}}\Big|_{t=t_0}$ but this is the reason it is the definition of T_H

(5) A big value of H

the distance to the sun (Appendix) is

$1 \text{ A.U.} = 1.50 \times 10^{13}$ cm

so the circumference of our orbit is $2\pi \times 1$ A.U.
The earth covers this distance in 1 year, so our velocity is

$$v = \frac{2\pi (1.5 \times 10^{13}) \text{ cm}}{365 \times 24 \times 60 \times 60 \text{ sec}} = 2.99 \times 10^{6} \ \frac{\text{cm}}{\text{sec}}$$

and we are professing that the sun is receding at $\frac{1}{100}$ this speed, or $2.99 \times 10^{4} \ \frac{\text{cm}}{\text{sec.}}$
setting this $= H \times 1$ A.U. we solve to find

$$H = \frac{2.99 \times 10^{4} \text{ cm/sec}}{1 \text{ AU}} = 1.99 \times 10^{-9} \text{ sec}^{-1}$$

$$\frac{1}{H} = 15.9 \text{ years } !$$