Math 111, Introduction to the Calculus, Fall 2011
Midterm III Solutions

1. *Calculate the following integrals*

   (a) $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\, dx$

   (b) $\displaystyle\int_{-\pi/2}^0 \sin(2t)\, dt$

   (a) Since $\frac{1}{\sqrt{x}} = x^{-1/2}$ we have

$$\begin{aligned}\int_1^4 x^{-1/2}\, dx &= \left[\frac{x^{1/2}}{1/2}\right]_{x=1}^{x=4}\\ &= 2(4^{1/2}) - 2(1^{1/2})\\ &= 4-2\\ &= 2\end{aligned}$$

   (b) We can guess that the antiderivative of $\sin(2t)$ involves $\cos(2t)$. In fact, the derivative of $\cos(2t)$ is

$$-2\sin(2t)$$

   so the antiderivative of $\sin(2t)$ is $-\frac{1}{2}\cos(2t)$. Therefore

$$\begin{aligned}\int_{-\pi/2}^0 \sin(2t)\, dt &= \left[-\frac{1}{2}\cos(2t)\right]_{t=-\pi/2}^{t=0}\\ &= (-\frac{1}{2}\cos(0)) - (-\frac{1}{2}\cos(-\pi))\\ &= -\frac{1}{2} - \frac{1}{2}\\ &= -1\end{aligned}$$

2. *Find the critical points of the function*

$$f(x) = x\sin(x) + \cos(x)$$

   that are in the range $-\pi \leq x \leq \pi$. Classify each critical point as either a local maximum, a local minimum or neither.

   To find the critical points we find $f'(x)$:

$$\begin{aligned}f'(x) &= \sin(x) + x\cos(x) - \sin(x)\\ &= x\cos(x)\end{aligned}$$

The critical points therefore satisfy $x\cos(x) = 0$. This means that either $x = 0$ or $\cos(x) = 0$. In the range $-\pi \le x \le \pi$ the solutions to $\cos(x) = 0$ are $x = -\pi/2$ and $x = \pi/2$. So the critical points are:

$$x = 0, \quad x = -\pi/2, \quad x = \pi/2.$$

To classify these, we use the second derivative test. We have

$$f''(x) = \cos(x) - x\sin(x).$$

Then:

$$f''(-\pi/2) = \cos(-\pi/2) + \pi/2\sin(-\pi/2) = -\pi/2 < 0,$$

so $x = -\pi/2$ is a local maximum;

$$f''(0) = \cos(0) = 1 > 0,$$

so $x = 0$ is a local minimum;

$$f''(\pi/2) = \cos(\pi/2) - \pi/2\sin(\pi/2) = -\pi/2 < 0,$$

so $x = \pi/2$ is a local maximum.

3. *For the function*

$$f(x) = \frac{x^2}{x+1}$$

we have

$$f'(x) = \frac{x(x+2)}{(x+1)^2}$$

and

$$f''(x) = \frac{2}{(x+1)^3}.$$

(a) Where does the graph of $f(x)$ have a vertical asymptote?

(b) On which intervals is $f(x)$ increasing or decreasing?

(c) On which intervals is $f(x)$ concave up or concave down?

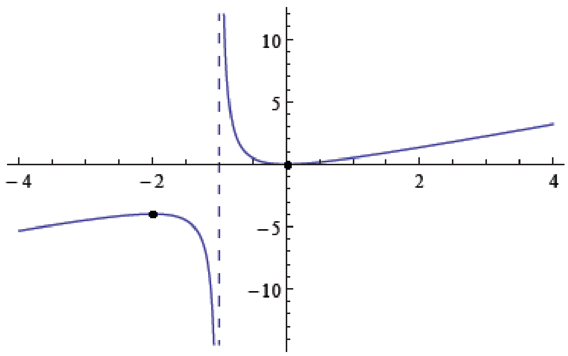
Sketch a graph of the function f for the range $-4 \le x \le 4$ that displays your answers to the above questions. (Your graph should also show that $f(0) = 0$ and $f(-2) = -4$. There is extra space on the next page for this question so you can make your graph nice and big.)

(a) $x = -1$

(b) Since $(x+1)^2$ is always positive, it is sufficient to look at the sign of $x(x+2)$.
For $x < -2$, $x < 0$ and $x + 2 < 0$, so $f'(x) > 0$ and f is increasing.
For $-2 < x < -1$, $x < 0$ and $x + 2 > 0$ so $f'(x) < 0$ and f is decreasing.
For $-1 < x < 0$, $x < 0$ and $x + 2 > 0$ so $f'(x) < 0$ and f is decreasing.
For $x > 0$, $x > 0$ and $x + 2 > 0$ so $f'(x) > 0$ and f is increasing.

(c) For $x < -1$, we have $(x+1)^3 < 0$ so f is concave down. For $x > -1$, $(x+1)^3 > 0$ so f is concave up.

The graph looks as follows:



4. *(a) Show how to calculate*

$$\int_{-1}^{2} |2x|\, dx$$

by finding the relevant area (not by finding antiderivatives). You should include a diagram and explain your answer fully.

(b) Suppose you know that

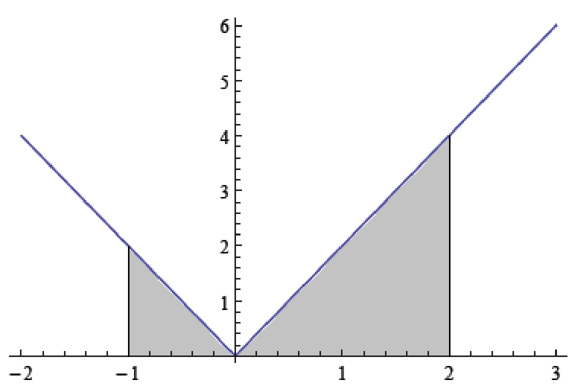
$$\int_0^2 g(x)\, dx = -4, \quad \int_0^4 g(x)\, dx = 5.$$

Find:

i. $\displaystyle\int_0^2 2g(x) - 1\, dx$

ii. $\displaystyle\int_2^4 g(x)\, dx$

(a) The graph of the function is



The integral is given by adding the two shaded areas. The one on the left is a triangle with base 1 unit and height 2 units, so its area is 1. The one of the right is a triangle with base 2 units and height 4 units, so its area is 4. Therefore the integral is equal to $1 + 4 = 5$.

(b) i. We have

$$\begin{aligned}\int_0^2 2g(x) - 1\,dx &= \int_0^2 2g(x)\,dx - \int_0^2 1\,dx \\ &= 2\int_0^2 g(x)\,dx - [x]_{x=0}^{x=2} \\ &= 2(-4) - 2 \\ &= -10\end{aligned}$$

ii. We have

$$\int_2^4 g(x)\,dx = \int_0^4 g(x)\,dx - \int_0^2 g(x)\,dx = 5 - (-4) = 9.$$

5. *If the sum of two positive numbers x and y is 10, find the maximum possible value of*

$$2y + x^2.$$

(As part of your answer, explain how you know there is a maximum value.)

[This question should have said that $x, y \geq 0$ so that they are allowed to be 0.]

We want to find the maximum value of $2y + x^2$ where $x + y = 10$. Then $y = 10 - x$ we are trying to maximize

$$f(x) = 2(10 - x) + x^2 = x^2 - 2x + 20.$$

The range of possible x-values is $0 \leq x \leq 10$. We have to test the critical points:

$$0 = f'(x) = 2x - 2$$

so that $x = 1$, where $f(1) = 1 - 2 + 20 = 19$. There are no points where f is not differentiable. We also have to test the endpoints $x = 0$ where $f(0) = 20$ and $x = 10$ where $f(10) = 100 - 20 + 20 = 100$.

Therefore the maximum value is 100 when $x = 10$ and $y = 0$.