Solutions to the Algebra problems on the Comprehensive Examination of January 27, 2012

1. **(25 points).** Let G be a group, and let $H, K \subseteq G$ be subgroups of G.

   (a) Prove the following standard theorem about subgroups: that $H \cap K$ is a subgroup of G.

   Solution: ($H \cap K$ nonempty) Since H and K are subgroups of G, $e \in H$ and $e \in K$ so $e \in H \cap K$, which means that $H \cap K \neq \emptyset$.✓
   ($H \cap K$ closed under group operation) Given $a, b \in H \cap K$, $ab \in H$ and $ab \in K$ since H and K are closed under $*$, so $ab \in H \cap K$ as desired.✓
   ($H \cap K$ closed under $^{-1}$) Given $a \in H \cap K$, $a^{-1} \in H$ and $a^{-1} \in K$ since H and K are closed under inverses, so $ab \in H \cap K$ as desired.✓
   Thus $H \cap K$ is a subgroup of G as desired. QED

   (b) If H and K are both *normal* subgroups of G, prove that $H \cap K$ is also a normal subgroup of G.

   Solution: First of all, we know from part (a) that $H \cap K$ is a subgroup of G. Now, given $a \in H \cap K$, $g \in G$, $gag^{-1} \in H$ and $gag^{-1} \in K$ since H and K are normal. Thus, $gag^{-1} \in H \cap K$, so $H \cap K$ is normal in G as desired. QED

2. **(25 points).** Let G and H be groups. Recall that a homomorphism $\phi : G \to H$ is said to be *trivial* if $\phi(g) = e_H$ for all $g \in G$.

   (a) If $|G| = 144$ and $|H| = 25$, prove that any homomorphism $\phi : G \to H$ is trivial.

   Solution: Given $g \in G$, since $o(g) \mid |G| = 144$, we have $g^{144} = e_G$; note that similarly, $(\phi(g))^{25} = e_H$. Since ϕ is a homomorphism, we have

   $$e_H = \phi(e_G) = \phi(g^{144}) = (\phi(g))^{144}.$$

   In any group, $h^m = e$ implies $o(h) \mid m$. Applying this to $(\phi(g))^{25} = (\phi(g))^{144} = e_H$ gives $o\big(\phi(g)\big) \mid 144$ and $o\big(\phi(g)\big) \mid 25$. Since $\gcd(144, 25) = \gcd(2^4 3^2, 5^2) = 1$, we must have $o\big(\phi(g)\big) = 1$, which means $\phi(g) = e_H$ as desired. QED

   (b) Let G be the cyclic group of order 2, and let H be the cyclic group of order 6. Give an example of a nontrivial homomorphism $\phi : G \to H$.

   Solution: Define $\phi : G \to H$ by $0 \mapsto 0$, $1 \mapsto 3$. G is small enough that we can confirm this is a homomorphism by hand: $\phi(0 \oplus 0) = 0 = 0 \oplus 0 = \phi(0) \oplus \phi(0)$, $\phi(0 \oplus 1) = 3 = 0 \oplus 3 = \phi(0) \oplus \phi(1)$, and $\phi(1 \oplus 1) = 0 = 3 \oplus 3 = \phi(1) \oplus \phi(1)$.✓

3. **(25 points).**

(a) List all elements of A_4, the alternating group of degree four, expressing each such element as a product of disjoint cycles.

Solution: A_4 consists of e, 3-cycles, and "double 2-cycles":

$$A_4 = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3), (1\ 2\ 3), (1\ 3\ 2),$$

$$(1\ 2\ 4), (1\ 4\ 2), (1\ 3\ 4), (1\ 4\ 3), (2\ 3\ 4), (2\ 4\ 3)\}$$

Note that $|A| = 12 = 4!/2$ as it should be.

(b) For each element you listed, say what its order is.

Solution: $o(e) = 1$, the order of each 3-cycle is 3, and the order of the double 2-cycles ((1 2)(3 4), (1 3)(2 4), and (1 4)(2 3)) is 2.

4. **(25 points).** Let R be a ring.

(a) Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.

Solution: $I \subseteq R$ is an ideal of R if $(I, +)$ is a subgroup of $(R, +)$ and $\forall x \in I,\ r \in R$, we have $xr \in I$ and $rx \in I$.

(b) Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$. You may assume that R is a ring under the operations of matrix addition and matrix multiplication.
Let $I = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Prove that I is an ideal of R.

Solution: First, we show that $(I, +)$ is a subgroup of $(R, +)$:
(I non-empty) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in I$, so $I \neq \emptyset$.✓
(I closed under $+$) Given $x = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$, $x + y = \begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix} \in I$.✓
(I closed under negatives) Given $x = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, $-x = \begin{bmatrix} -a & -b \\ 0 & 0 \end{bmatrix} \in I$.✓
Thus $(I, +)$ is a subgroup of $(R, +)$.✓
Now, given $r = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R$ and $x = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$, $rx = \begin{bmatrix} ax & by \\ 0 & 0 \end{bmatrix} \in I$ and $xr = \begin{bmatrix} ax & bx + cy \\ 0 & 0 \end{bmatrix} \in I$.✓
Thus I is an ideal of R. QED