Solutions to the Analysis problems on the Comprehensive Examination of January 29, 2010

1. (10 points)

   (a) State the Axiom of Completeness (also known as the Axiom of Continuity for the Real Numbers or Axiom C).

   Solution: Every nonempty set of real numbers that is bounded above has a least upper bound.

   (b) State the Heine-Borel Theorem.

   Solution: A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

2. (10 points) Use induction to prove that

$$\sum_{k=1}^{n} kx^{k-1} = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$$

for all positive integers n.

Solution: When $n = 1$,

$$\sum_{k=1}^{n} kx^{k-1} = 1 \cdot x^0 = 1 \text{ and } \frac{1-2x+x^2}{(1-x)^2} = 1.$$

Hence the equation holds when $n = 1$.

Suppose the equation holds for some $n \in \mathbb{N}$. Then

$$\sum_{k=1}^{n} kx^{k-1} = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}.$$

Hence

$$\begin{aligned}
\sum_{k=1}^{n+1} kx^{k-1} &= \sum_{k=1}^{n} kx^{k-1} + (n+1)x^n \\
&= \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2} + (n+1)x^n \quad \text{(inductive hypothesis)} \\
&= \frac{1-(n+1)x^n + nx^{n+1} + (1-x)^2(n+1)x^n}{(1-x)^2} \\
&= \frac{1-(n+1)x^n + nx^{n+1} + (n+1)x^n - 2(n+1)x^{n+1} + (n+1)x^{n+2}}{(1-x)^2} \\
&= \frac{1-(n+2)x^{n+1} + (n+1)x^{n+2}}{(1-x)^2}.
\end{aligned}$$

Hence the equation still holds for $n+1$. Therefore

$$\sum_{k=1}^{n} kx^{k-1} = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$$

holds for all positive integers n.

Note: The problem should have stated that x is a real number different from 1.

3. (10 points)

   (a) State the definition of Cauchy sequence in $\mathbb{R}$
   Solution: A sequence (a_n) is called a Cauchy sequence if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq N$, it follows that $|a_m - a_n| < \epsilon$.

   (b) Use the definition of Cauchy sequence to prove that every Cauchy sequence in $\mathbb{R}$ is bounded.
   Solution: Suppose (a_n) is a Cauchy sequence. By definition, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|a_m - a_n| < 1$. Pick $n = N$, we get that there exists an $N \in \mathbb{N}$ satisfying for all $m \geq N, |a_m - a_N| < 1$. By the triangle inequality, this implies

$$|a_m| - |a_N| \leq |a_m - a_N| < 1$$

   for $m \geq N$. Thus $|a_m| < |a_N| + 1$ for $m \geq N$.
   Let $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}$. Then $|a_k| \leq M$ for all $k \in \mathbb{N}$. By definition, the sequence is bounded by M.

4. (10 points) Let $f_n(x) = \frac{nx}{1+nx}$ for $x \geq 0$.

   (a) State the function f to which the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise.
   Solution: First observe that $f_n(0) = 0$, so that $f_n(0)$ converges to 0. When $x > 0$, notice that $f_n(x) = 1 - \frac{1}{1+nx}$ and $\frac{1}{1+nx} \to 0$ as $n \to \infty$ since $x > 0$. Hence, for $x \geq 0$, $f_n(x)$ converges pointwise to

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0. \end{cases}$$

   (b) Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[1, \infty)$.
   Solution: Note that $f = 1$ on $[1, \infty)$. Take $\epsilon > 0$.
   If $\epsilon < 1$, we have $1 < \frac{1}{\epsilon}$. Pick $N \in \mathbb{N}$ with $N > \frac{1}{\epsilon} - 1$. Then for all $n \geq N, x \geq 1$, we have $|f_n(x) - f(x)| = |-\frac{1}{1+nx}| = \frac{1}{1+nx} \leq \frac{1}{1+n} \leq \frac{1}{1+N} < \frac{1}{1+(\frac{1}{\epsilon}-1)} = \epsilon$.
   If $\epsilon \geq 1$, let $N = 1$. For any $n \geq N, x \geq 1$. Then $|f_n(x) - f(x)| = |-\frac{1}{1+nx}| = \frac{1}{1+nx} \leq \frac{1}{1+1\cdot 1} < 1 \leq \epsilon$.
   Hence for each $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that $n \geq N$ gives $|f_n(x) - f(x)| < \epsilon$ for all $x \geq 1$. Then by definition $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[1, \infty)$.

(c) Explain why $\{f_n\}_{n=1}^{\infty}$ does NOT converge uniformly on $[0, \infty)$.

Solution 1: Note that each f_n is continuous on $[0, \infty)$. If convergence were uniform, then the limit function f would be continuous by a standard theorem in analysis. However, the formula for f given in (a) makes it clear that f is not continuous at 0. Hence convergence cannot be uniform.

Solution 2: To show the statement is false, we need to find one bad ϵ.

Let $\epsilon = 1/2$, and for each $N \in \mathbb{N}$, pick $x = \frac{1}{N}$. Then we have $|f_N(x) - f(x)| = |\frac{N(\frac{1}{N})}{1+N(\frac{1}{N})}| = \frac{1}{2}$. Hence no $N \in \mathbb{N}$ can satisfy the condition that $|f_n(x) - f(x)| < 1/2$ for all $x \in [0, \infty)$, $n \geq N$.

Therefore $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly on $[0, \infty)$.