**Solutions to the Algebra problems on the Comprehensive Examination of February 1, 2013**

1. **(25 points).** Let $G$ be a group, let $H \subseteq G$ be a subgroup, and let $N \triangleleft G$ be a **normal** subgroup. Define
$$NH = \{xh : x \in N \text{ and } h \in H\}.$$
Prove that $NH$ is a subgroup of $G$.

**Solution:** We must show that $NH$ is (1) nonempty, (2) closed under multiplication, and (3) closed under inverses.

(1) There exist $x \in N$ and $h \in H$, since $N, H \neq \varnothing$. So $xh \in NH$, and so $NH \neq \varnothing$.

(2) Given arbitrary $x_1h_1, x_2h_2 \in NH$ with $x_1, x_2 \in N$, and $h_1, h_2 \in H$, we would like to show that $(x_1h_1)(x_2h_2) \in NH$. Note that $h_1x_2h_1^{-1} \in N$, since $N \triangleleft G$. So
$$(x_1h_1)(x_2h_2) = x_1h_1x_2(h_1^{-1}h_1)h_2 = \big(x_1(h_1x_2h_1^{-1})\big)\big(h_1h_2\big) \in NH,$$
since $x_1(h_1x_2h_1^{-1}) \in N$ and $h_1h_2 \in H$, by closure under multiplication.

(3) Given arbitrary $xh \in NH$ with $x \in N$ and $h \in H$, we would like to show that $(xh)^{-1} \in NH$. Note that $x^{-1} \in N$, and hence $h^{-1}x^{-1}h \in N$, since $N \triangleleft G$. Thus,
$$(xh)^{-1} = h^{-1}x^{-1} = (h^{-1}x^{-1})(hh^{-1}) = (h^{-1}x^{-1}h)h^{-1} \in NH,$$
since $h^{-1} \in H$ because $H$ is closed under inverses. QED

**Alternate Proofs of (2) and (3)**:

(2) Given arbitrary $x_1h_1, x_2h_2 \in NH$ with $x_1, x_2 \in N$, and $h_1, h_2 \in H$, we would like to show that $(x_1h_1)(x_2h_2) \in NH$. Since $N \triangleleft G$, we have
$$h_1x_2 \in h_1N = Nh_1,$$
and hence there exists $y \in N$ such that $h_1x_2 = yh_1$. Thus,
$$(x_1h_1)(x_2h_2) = x_1(h_1x_2)h_2 = x_1(yh_1)h_2 = (x_1y)(h_1h_2) \in NH,$$
since $x_1y \in N$ and $h_1h_2 \in H$ by closure under multiplication.

(3) Given arbitrary $xh \in NH$ with $x \in N$ and $h \in H$, we would like to show that $(xh)^{-1} \in NH$. Since $x^{-1} \in N$ and $N \triangleleft G$, we have
$$h^{-1}x^{-1} \in h^{-1}N = Nh^{-1},$$
and hence there exists $y \in N$ such that $h^{-1}x^{-1} = yh^{-1}$. Thus,
$$(xh)^{-1} = h^{-1}x^{-1} = yh^{-1} \in NH,$$
since $h^{-1} \in H$ because $H$ is closed under inverses. QED

2. **(25 points).** Let $G_1$ and $G_2$ be groups, let $\phi : G_1 \to G_2$ be a homomorphism, and let $H_1 \subseteq G_1$ be a subgroup. Recall that the set

$$H_2 = \{\phi(x) : x \in H_1\}$$

is a subgroup of $G_2$, called the *image of* $H_1$ *under* $\phi$, sometimes notated $\phi(H_1)$.

(a) (10 points). If $G_2$ is finite, prove that $|H_2| \Big| |G_2|$.
That is, prove that the order of $H_2$ divides the order of $G_2$.
**Solution:** Since $G_2$ is a finite group with subgroup $H_2$, Lagrange's Theorem says that $|H_2| \Big| |G_2|$. QED

(b) (15 points). If $G_1$ is finite, prove that $|H_2| \Big| |G_1|$.
That is, prove that the order of $H_2$ divides the order of $G_1$.
**Solution:** Let $K$ denote the kernel of $\phi$. Then by the fundamental theorem of group homomorphisms, $\phi(G_1) \cong G_1/K$, which implies that $|\phi(G_1)| = |G_1/K|$. Furthermore, because $G_1$ is also finite, Lagrange's Theorem tells us that

$$|G_1| = [G_1 : K] \cdot |K| = |G_1/K| \cdot |K| = |\phi(G_1)| \cdot |K|.$$

Since $|K|$ is an integer, we have shown that $|\phi(G_1)|$ divides $|G_1|$. Moreover, since $H_1 \subseteq G_1$, we have

$$H_2 = \{\phi(x) : x \in H_1\} \subseteq \phi(G_1) = \{\phi(x) : x \in G_1\}.$$

Again by Lagrange's Theorem, then, $|H_2|$ divides $|\phi(G_1)|$. Therefore $|H_2|$ divides $|G_1|$, because divisibility is transitive. QED

3. **(25 points).** Consider the group $S_{100}$ of permutations of the set $\{1, 2, 3, \dots, 100\}$. Let $\sigma \in S_{100}$ be the permutation

$$\sigma = (3\ 6\ 4)(1\ 5\ 2\ 4)(1\ 6\ 5\ 3\ 2).$$

(a) (7 points). Write $\sigma$ as a product of **disjoint** cycles.
**Solution:** $\sigma = (1\ 4)(2\ 5\ 6)$.

(b) (7 points). Compute the **order** of $\sigma$.
**Solution:** Since $\sigma$ is a product of disjoint cycles of length 2 and 3, we have $o(\sigma) = \operatorname{lcm}(2, 3) = 6$.

(c) (8 points). For each integer $n = 7, 8, \dots, 100$, let $\tau_n = (1\ n\ 5)$. For each such $n$, decide whether the product $\sigma\tau_n$ is an **even** or **odd** permutation.
**Solution**: For each $n = 7, \dots, 100$, $\tau_n$ is a 3-cycle and hence is an even permutation (since 3 is odd). Meanwhile, $\sigma$ is a product of a 2-cycle (odd) and a 3-cycle (even). So $\sigma\tau_n$ is

$$\text{odd} + \text{even} + \text{even} = \text{odd}.$$

4. **(25 points).** Let $R$ be a ring.

(a) (10 points). Define what it means for a subset $I \subseteq R$ to be an **ideal** of $R$. If you use any other technical terms like "closed," "subring," "subgroup," "coset," etc., you must fully define those terms as well.

**Solution:** $I \subseteq R$ is said to be an ideal of $R$ if

i. $I$ is nonempty;

ii. for every $x$, $y \in I$, we have $x - y \in I$; and

iii. for every $a \in R$ and $b \in I$, we have $ab$, $ba \in I$.

(b) (15 points). Suppose that $R$ is commutative and has a multiplicative identity 1. Let $I \subseteq J \subseteq R$ be ideals, and suppose that the quotient ring $R/I$ is a field.

If $I \subsetneq J$, prove that $1 \in J$.

[In fact, it is a Theorem from Math 350 that $J = R$ in this case, but you are only being asked to prove that $1 \in J$. In particular, however, you may **not** quote the $J = R$ theorem.]

**Solution:** Recall that $R/I$ has zero element $I + 0$ and multiplicative identity $I + 1$. [Like all elements of $R/I$, they are cosets of $I$.]

Since $I \subsetneq J$, there exists $r \in J \setminus I$. This implies $r \notin I$, so that

$$I + r \neq I + 0$$

by the coset criterion. Since $R/I$ is a field, every nonzero element of $R/I$ has a multiplicative inverse, and hence there is some $I + s \in R/I$ such that

$$(I + r)(I + s) = I + 1.$$

So $I + rs = I + 1$, and hence $1 - rs \in I$ by the coset criterion. Thus,

$$1 - rs \in I \subseteq J.$$

But $J$ is an ideal of $R$ and $s \in R$, so $rs \in J$. Therefore, since $J$ is closed under addition, we have

$$1 = (1 - rs) + rs \in J.$$

QED