Solutions to the Analysis problems on the Comprehensive Examination of February 1, 2013

1. (a) (2 points) What does it mean for a sequence (a_n) of real numbers to be *bounded*?

   Solution: (a_n) is said to be bounded if there exists a positive $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

   (b) (2 points) State the Bolzano-Weierstrass Theorem as it applies to a bounded sequence (a_n) of real numbers.

   Solution: Every bounded sequence of real numbers (a_n) contains a convergent subsequence.

2. Consider the sequence $(f_n)_{n \geq 1}$ of functions where $f_n : [0, \infty) \to \mathbb{R}$ is defined by

$$f_n(x) = \begin{cases} 1 - nx & \text{for } 0 \leq x \leq \frac{1}{n} \\ 0 & \text{for } x > \frac{1}{n}. \end{cases}$$

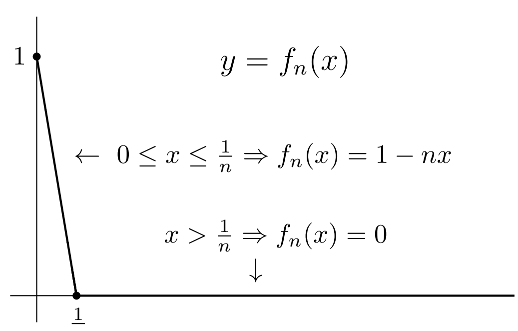
   (a) (8 points) Prove that (f_n) converges pointwise to a function f and give an explicit description of f.

   Solution: First observe that $f_n(0) = 1 - n \cdot 0 = 1$ for all n, so that $f_n(0) \to 1$ as $n \to \infty$. Next assume $0 < x$ and pick N such that $\frac{1}{N} < x$. For $n \geq N$, have $\frac{1}{n} \leq \frac{1}{N} < x$, so that $f_n(x) = 0$ for $n \geq N$. So for all $x > 0$, $f_n(x) \to 0$ as $n \to \infty$. It follows that (f_n) converges pointwise on $[0, \infty)$ to

$$f(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x \in (0, \infty). \end{cases}$$

   (b) (6 points) Prove that (f_n) does not converge uniformly to f.

   Solution: For $n \in \mathbb{N}$, we draw the graph of f_n:



   This makes it clear that f_n is continuous on $[0, \infty)$ for $n \in \mathbb{N}$. However, the limit function $f(x)$ is clearly discontinuous at 0. Hence, since uniform convergence preserves continuity (this is a standard theorem in analysis), (f_n) does not converge uniformly to f on $[0, \infty)$.

3. (a) (4 points) State the Cauchy Criterion for a series $\sum_{n=1}^{\infty} a_n$ of real numbers to converge.

Solution: $\sum_{n=1}^{\infty} a_n$ converges if and only if, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > m \geq N$, then $|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon$.

(b) (8 points) Suppose that a series $\sum_{n=1}^{\infty} a_n$ of real numbers converges absolutely. Prove that the series converges.

Solution: Let $\epsilon > 0$ be given. Because $\sum_{n=1}^{\infty} |a_n|$ converges, by the Cauchy Criterion stated above, there exists $N \in \mathbb{N}$ such that $\big||a_{m+1}|+|a_{m+2}|+\cdots+|a_n|\big| = |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \epsilon$ for all $n > m \geq N$. By the triangle inequality, $|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n|$. Since the latter is $< \epsilon$, we get $|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon$. Using the Cauchy Criterion again, we conclude that $\sum_{n=1}^{\infty} a_n$ also converges.

4. (10 points) For this question, do EITHER part (a) OR part (b), NOT BOTH.

(a) Prove that the function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not Riemann-integrable.

Solution: Let $\mathcal{P}$ be the collection of all possible partitions of $[0,1]$. Fix an arbitrary $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\} \in \mathcal{P}$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, every subinterval $[x_{k-1}, x_k]$ of P will contain a point $x \in \mathbb{Q}$ where $f(x) = 1$. It follows that $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} = 1$, so

$$U(f,P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} x_k - x_{k-1} = 1.$$

Thus, because P is arbitrary, the upper integral of f is

$$U(f) = \inf\{U(f,P) : P \in \mathcal{P}\} = 1.$$

By a similar argument, this time utilizing the fact that the irrationals are dense in $\mathbb{R}$, we see that $L(f,P) = 0$ for all $P \in \mathcal{P}$. Therefore the lower integral of f is

$$L(f) = \sup\{L(f,P) : P \in \mathcal{P}\} = 0.$$

It follows that $U(f) \neq L(f)$, so by definition f is not Riemann-integrable.

(b) Prove that a nonempty compact set K of real numbers has a maximum element: that is, show that there is $x \in K$ such that $x \geq y$ for all $y \in K$.

Solution: Since K is compact, by Heine-Borel it is bounded, which means it is bounded above, so because it is also nonempty, by the Axiom of Completeness, $\sup(K)$ exists. Call it x. By the definition of supremum, $x \geq y\ \forall\ y \in K$. So it remains to show that $x \in K$. The idea is to assume $x \notin K$ and derive a contradiction. There are two ways to do this.

Use open sets. Note that $x \in \mathbb{R} \setminus K$, which is open because K is closed. By the definition of open, there exists an ϵ-neighborhood

$$(\star) \qquad V_\epsilon(x) = (x - \epsilon, x + \epsilon) \subseteq \mathbb{R} \setminus K.$$

We show that $x - \epsilon$ is also an upper bound of K as follows. Given $k \in K$, we have $k \leq x$. Since $(\star)$ implies $k \notin (x - \epsilon, x + \epsilon)$, this forces $k \leq x - \epsilon$. Thus $x - \epsilon$ is an upper bound of K, which contradicts the assumption that $x = \sup(K)$.

Use limit points. Let $\epsilon > 0$. Since $x = \sup(K)$, we know that $x - \epsilon$ is not a upper bound of K. Hence there exists $k \in K$ such that $x - \epsilon < k$. Because $k \leq x$, $k \in (x - \epsilon, x + \epsilon) = V_\epsilon(x)$. Since we are assuming $x \notin K$, we also have $k \neq x$. Hence $k \in V_\epsilon(x) \setminus \{x\}$. Thus we have proved that every ϵ-neighborhood of x intersects K in some point other than x, which means that x is a limit point of K. By definition, a closed set contains its limit points, and K is closed by Heine-Borel. Hence $x \in K$, which contradicts our assumption that $x \notin K$.