Solutions to the Multivariable Calculus and Linear Algebra problems on the Comprehensive Examination of February 1, 2013

Each question is worth 10 points, totaling 80 points.

1. Evaluate the following integrals:

   (a) $\iint_R (x+y)\,dy\,dx$, where R is the top half of the circle of radius 2 centered at the origin.

   Solution: In polar coordinates, $R = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le \pi\}$. Then

$$\begin{aligned}
\iint_R (x+y)\,dy\,dx &= \int_0^{\pi}\int_0^2 (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta \\
&= \int_0^{\pi}\int_0^2 (\cos\theta + \sin\theta)\,r^2\,dr\,d\theta \\
&= \int_0^{\pi} (\cos\theta + \sin\theta)\left[\frac{r^3}{3}\right]_0^2 d\theta \\
&= \frac{8}{3}\int_0^{\pi} (\cos\theta + \sin\theta)\,d\theta \\
&= \frac{8}{3}\Big[\sin\theta - \cos\theta\Big]_0^{\pi} \\
&= \frac{8}{3}\big((\sin\pi - \cos\pi) - (\sin 0 - \cos 0)\big) \\
&= \frac{8}{3}\big((0-(-1)) - (0-1)\big) = \frac{16}{3}.
\end{aligned}$$

   (b) $\int_C (2xy + \tan(x^3))\,dx + \big(x^2 + 2xy\big)\,dy$, where C is the closed curve that begins at $(0,0)$, then follows $y = x^2$ to $(1,1)$, next follows $y = 1$ to $(0,1)$, and finally follows $x = 0$ back to $(0,0)$.

   Solution: We could divide the piecewise-smooth curve C into its three smooth segments and calculate the integral piece by piece, but it is far easier to use Green's Theorem. Let $P = 2xy + \tan(x^3)$ and $Q = x^2 + 2xy$. Then we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (2x + 2y) - 2x = 2y.$$

   We may express the region bounded by C as $R = \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le 1\}$. Then by Green's Theorem,

$$\int_C P\,dx + Q\,dy = \iint_R 2y\;dA = \int_0^1\int_{x^2}^1 2y\,dy\,dx$$

$$= \int_0^1 \left[y^2\right]_{x^2}^1 \, dx$$

$$= \int_0^1 (1 - x^4) \, dx$$

$$= x - \frac{x^5}{5}\Big|_0^1 = \frac{4}{5}.$$

2. Compute the volume of the 3-dimensional region that lies above the paraboloid $z = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 2$. Hint: Try cylindrical coordinates.

   Solution: The surfaces $z = x^2 + y^2$ and $x^2 + y^2 + z^2 = 2$ intersect where $z + z^2 = 2$, so $0 = z^2 + z - 2 = (z-1)(z+2)$. Since $z = x^2 + y^2$ cannot be negative, we have $z = 1$.

   In cylindrical coordinates the paraboloid is $z = r^2$, and the sphere is $r^2 + z^2 = 2$, or $z = \sqrt{2 - r^2}$. Thus we may express the region as

$$E = \{(r,\, \theta,\, z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ r^2 \le z \le \sqrt{2 - r^2}\}.$$

   Therefore the volume is

$$V = \iiint_E dV = \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^1 \left(r\sqrt{2 - r^2} - r^3 \right) dr$$

$$= 2\pi \left[-\frac{1}{3}\left(2 - r^2\right)^{\frac{3}{2}} - \frac{1}{4} r^4 \right]_0^1$$

$$= 2\pi \Big(\big(-\frac{1}{3} - \frac{1}{4} \big) - \big(-\frac{1}{3} \cdot 2^{3/2} - 0 \big) \Big)$$

$$= \frac{8\sqrt{2} - 7}{6} \pi.$$

3. Consider the function

$$f(x, y) = \begin{cases} \dfrac{x^4}{x^2 + y^2} & \text{if } (x, y) \ne (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

   (a) Show that f is continuous at $(0, 0)$.

   Solution: Recall that f is continuous at $(0,0)$ iff $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$. In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, and $(x, y) \to (0, 0)$ becomes $r \to 0$. Then

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2 + y^2} = \lim_{r\to 0} \frac{r^4 \cos^4\theta}{r^2} = \lim_{r \to 0} r^2 \cos^4\theta = 0,$$

   where the last equality follows since $r^2 \to 0$ and $\cos^4\theta - \sin^4\theta$ is bounded.

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

Solution: We use the definition of partial derivatives.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^4}{h^2+0} - 0}{h} = \lim_{h\to 0} h = 0,$$

and

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

4. Find the maximum and minimum values of the function $f(x,y) = x^2 + 4y$ subject to the constraint $x^2 + 2y^2 = 8$.

Solution: Let $g(x,y) = x^2 + 2y^2$. Then we need to find values of x, y and λ such that $\nabla f = \lambda \nabla g$ and $g(x,y) = 8$. We have the following system of equations:

$$2x = \lambda(2x), \quad 4 = \lambda(4y), \quad x^2 + 2y^2 = 8.$$

Writing the first equation as $2x(1-\lambda) = 0$ yields either $x = 0$ or $\lambda = 1$. When $x = 0$, the third equation yields $y = \pm 2$; when $\lambda = 1$, the second equation yields $y = 1$, so the third equation gives $x = \pm\sqrt{6}$. Thus we get the points $(x,y) = (0,\pm 2), (\pm\sqrt{6},1)$. Then we compute

$$f(0,2) = 8, \quad f(0,-2) = -8, \quad f(\pm\sqrt{6},1) = 10.$$

We conclude the maximum value of f is 10 at $\left(\pm\sqrt{6},1\right)$ and the minimum value of f is -8 at $(0,-2)$.

5. Let A be a 5×7 matrix with real entries. Answer the following questions about A and briefly justify your answers:

(a) Show that the nullspace of A has dimension at least 2.

Solution: By the rank/nullity theorem,

$$\text{rank}\, A + \text{nullity}\, A = \text{the number of columns of } A = 7.$$

The nullity of A is the dimension of its nullspace, and the rank of A is equal to its maximum number of linearly independent rows. Since A has 7 columns and 5 rows, the rank is at most 5, so its nullity is at least $7-5=2$ (because A may have fewer than 5 linearly independent rows).

(b) Assume that the nullspace has dimension exactly 2. What does this imply about the column space of A?

Solution: For reasons given above we know that $\text{rank}\, A = 5$. Thus we also know that the dimension of the column space of A is 5. This means that five of the column vectors of A are linearly independent and form a basis of the column space of A. Further, since the column space of A is a 5-dimensional subspace of $\mathbb{R}^5$, we can conclude that the column space of A is all of $\mathbb{R}^5$

6. Let U and W be subspaces of a vector space V. Define

$$U + W = \{u + w \mid u \in U,\ w \in W\}.$$

(a) Prove that $U + W$ is a subspace of V.

Solution: To prove that $U + W$ is a subspace, we must show that it is (1) nonempty, (2) closed under addition, and (3) closed under scalar multiplication. For (1), since U and W both contain $\vec{0}$, so does $U+W$ and so $U+W$ is nonempty;

(2) Let $\vec{v}_1, \vec{v}_2 \in U + W$. Then $\vec{v}_1 = \vec{u}_1 + \vec{w}_1$ and $\vec{v}_2 = \vec{u}_2 + \vec{w}_2$, where $\vec{u}_1, \vec{u}_2 \in U$ and $\vec{w}_1, \vec{w}_2 \in W$. So

$$\vec{v}_1 + \vec{v}_2 = (\vec{u}_1 + \vec{w}_1) + (\vec{u}_2 + \vec{w}_2) = (\vec{u}_1 + \vec{u}_2) + (\vec{w}_1 + \vec{w}_2).$$

But $\vec{u}_1 + \vec{u}_2 \in U$ because U is a subspace, and similarly $\vec{w}_1 + \vec{w}_2 \in W$. Thus $\vec{v}_1 + \vec{v}_2 \in U + W$, so $U + W$ is closed under addition;

(3) Let k be a scalar. Then $k\vec{v}_1 = k(\vec{u}_1 + \vec{w}_1) = k\vec{u}_1 + k\vec{w}_1$. But $k\vec{u}_1 \in U$ and $k\vec{w}_1 \in W$, so $k\vec{v}_1 \in U + W$, so $U + W$ is closed under scalar multiplication.

(b) For the vector space $\mathbb{R}^2$, give an explicit example of subspaces $U \subseteq \mathbb{R}^2$ and $W \subseteq \mathbb{R}^2$ such that $U \cup W$ is not a subspace of $\mathbb{R}^2$. Justify your reasoning.

Solution: There are many solutions to this problem. Here is one. Let $U = \text{Span}\{(1,0)\}$ be the x-axis and $W = \text{Span}\{(0,1)\}$ be the y-axis. Being spans, U and W are subspaces of $\mathbb{R}^2$. Then $(1,0), (0,1) \in U \cup W$, but

$$(1,0) + (0,1) = (1,1) \notin U \cup W.$$

So $U \cup W$ is not closed under addition and is therefore not a subspace of $\mathbb{R}^2$.

7. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Determine the eigenvalues of A and a basis of each eigenspace of A.

Solution: The eigenvalues of A are the roots of its characteristic polynomial

$$\begin{aligned}
\det(A - \lambda I_3) &= \det\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 0 & 0 & -\lambda \end{bmatrix} \\
&= -\lambda \det(\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = -\lambda((1-\lambda)^2 - 1) \\
&= -\lambda(1 - 2\lambda + \lambda^2 - 1) = -\lambda(-2\lambda + \lambda^2) = -\lambda^2(\lambda - 2).
\end{aligned}$$

Hence the characteristic polynomial has two roots: $\lambda = 0$ and $\lambda = 2$, which are the two eigenvalues of A.

Next we compute the corresponding eigenvectors.

$\lambda = 0$: The eigenspace is the solution space of $(A - 0I)\,\vec{x} = \vec{0}$. Being homogeneous, we need only row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so we have one equation: $x_1 + x_2 + x_3 = 0$. Thus x_2 and x_3 are free variables. Solving for x_1 gives $x_1 = -x_2 - x_3$, so the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, the eigenspace of A associated to $\lambda = 0$ has basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$\lambda = 2$: Row reducing the coefficient matrix of $(A - 2I)\,\vec{x} = \vec{0}$ gives

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which gives the system $x_1 - x_2 = x_3 = 0$ with free variable x_2. The solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Thus the eigenspace of A associated to $\lambda = 2$ has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(b) Is A diagonalizable? Explain your reasoning.

Solution: A is diagonalizable because it is a 3×3 matrix and has three linearly independent eigenvectors.

8. Let $T : V \to W$ be a linear transformation from the vector space V to the vector space W. Assume that $v_1, ..., v_n \in V$ have the property that $T(v_1), ..., T(v_n)$ are linearly independent.

   (a) Prove that $v_1, ..., v_n$ are also linearly independent.

   Solution: We aim to prove that the only linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ that equals $\vec{0}_V$ is the trivial one with all coefficients equal to zero. Suppose $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}_V$. Because T is a linear transformation,

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}_V \quad \Rightarrow$$

$$\begin{aligned} T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = \vec{0}_W \quad &\Rightarrow \\ T(c_1\vec{v}_1) + T(c_2\vec{v}_2) + \cdots + T(c_n\vec{v}_n) = \vec{0}_W \quad &\Rightarrow \\ c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_nT(\vec{v}_n) = \vec{0}_W \quad &\Rightarrow \\ c_1 = \cdots = c_n = 0, & \end{aligned}$$

where the last implication follows since $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ are linearly independent.

(b) Assume in addition that $v_1, \ldots, v_n$ span V. Prove that T is one-to-one (injective).

Solution: By a standard theorem in linear algebra, it suffices to prove that the null space of T consists of only the zero vector. So take $\vec{u} \in V$ such that $T(\vec{u}) = \vec{0}_W$. Since $\vec{v}_1, \ldots, \vec{v}_n$ span V, we can write $\vec{u} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. Then

$$\vec{0}_W = T(\vec{u}) = T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + \cdots + c_nT(\vec{v}_n),$$

so that $c_1 = \cdots = c_n = 0$ since $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ are linearly independent. It follows that $\vec{u} = 0 \cdot \vec{v}_1 + \cdots + 0 \cdot \vec{v}_n = \vec{0}_V$, and one-to-one follows.