**Solutions to the Algebra problems on the Comprehensive Examination of January 31, 2014**

1. **(20 points).** Let $G$ and $H$ be groups, let $\phi : G \to H$ be a homomorphism, and suppose $g \in G$ is an element of some finite order $n \geq 1$.

   (a) (10 points). Show that that order of $\phi(g)$ divides $n$.

   **Solution:** Because $g$ has order $n$, we have $g^n = e_G$, and hence

   $$\big(\phi(g)\big)^n = \phi(g^n) = \phi(e_G) = e_H.$$

   Therefore, $o(\phi(g))$ divides $n$. QED

   (b) (10 points). Suppose that $|G| = 200$, $|H| = 72$, and the chosen element $g \in G$ has order $n = 25$. Prove that $g$ belongs to the kernel of $\phi$.

   **Solution:** We will compute $o(\phi(g))$. Since $g^{25} = e_G$, it follows from part (a) that

   $$o\big(\phi(g)\big)\big|25.$$

   But $\phi(g) \in H$, so because $H$ is finite, we have

   $$o\big(\phi(g)\big)\Big||H| = 72,$$

   by Lagrange's Theorem. Since $\gcd(25, 72) = 1$, we have $o(\phi(g)) = 1$. Thus, $\phi(g) = e_H$, and hence $g \in \ker(\phi)$. QED

2. **(30 points).** Consider the group $S_{10}$ of permutations of the set $\{1, 2, 3, ..., 10\}$. Let $\sigma, \tau \in S_{10}$ be the permutations

   $$\sigma = (1, 2, 3)(4, 5, 6) \quad \text{and} \quad \tau = (3, 4)(2, 7, 8, 5).$$

   (a) (6 points). Write $\sigma\tau$ as a product of **disjoint** cycles.

   **Solution:** $\sigma\tau = (1, 2, 3)(4, 5, 6)(3, 4)(2, 7, 8, 5) = (3, 5)(1, 2, 7, 8, 6, 4)$.

   (b) (12 points). Compute the **order** of each of $\sigma$, $\tau$, and $\sigma\tau$.

   **Solution:** Using the LCM formula for order of a permutation given its disjoint cycle decomposition, we have

   $$o(\sigma) = \operatorname{lcm}(3, 3) = 3, \qquad o(\tau) = \operatorname{lcm}(2, 4) = 4, \qquad o(\sigma\tau) = \operatorname{lcm}(2, 6) = 6.$$

   (c) (12 points). Decide whether each of $\sigma$, $\tau$, and $\sigma\tau$ is an **even** or **odd** permutation; don't forget to justify.

   **Solution:** $\sigma$ is a product of two 3-cycles (both even, since 3 is odd), so

   $$\sigma \text{ is:} \quad \text{even} + \text{even} = \text{even}.$$

   Similarly,

   $$\tau \text{ is:} \quad \text{odd} + \text{odd} = \text{even},$$

   so $\sigma\tau$ is the product of two evens and hence

   $$\sigma\tau \text{ is:} \quad \text{even} + \text{even} = \text{even}.$$

3. **(25 points).** Let $R$ be a ring.

(a) (10 points) Define what it means for a subset $I \subseteq R$ to be an **ideal** of $R$. If you use any other technical terms like "closed," "subring," "subgroup," "coset," etc., you must fully define those terms as well.

**Solution:** $I \subseteq R$ is an ideal of $R$ if

i. $I$ is nonempty,

ii. for every $x, y \in I$, we have $x - y \in I$, and

iii. for every $a \in R$ and $b \in I$, we have $ab, ba \in I$.

(b) (15 points) For the polynomial ring $R = \mathbb{R}[x]$, define

$$I = \{f \in R : f(2) = f(5) = 0\}.$$

Prove that $I$ is an ideal of $R$.

**Solution:** We check each of the three criteria listed above.

i. The constant polynomial $0 \in R$ satisfies $0(2) = 0(5) = 0$, so $0 \in I$.

ii. Given $f, g \in I$, we have $f(2) = f(5) = g(2) = g(5) = 0$. So

$$(f-g)(2) = f(2)-g(2) = 0-0 = 0, \quad \text{and} \quad (f-g)(5) = f(5)-g(5) = 0-0 = 0.$$

So $f - g \in I$.

iii. Given $f \in R$ and $g \in I$, we have $g(2) = g(5) = 0$. So

$$(fg)(2) = f(2)g(2) = f(2) \cdot 0 = 0, \quad \text{and} \quad (fg)(5) = f(5)g(5) = f(5) \cdot 0 = 0.$$

So $fg \in I$. In addition, $gf = fg$, so $gf \in I$. QED

4. **(25 points).** A nonzero element $a$ of a ring is said to be *nilpotent* if there is a positive integer $n \geq 1$ such that $a^n = 0$. (The element 0 itself is *not* said to be nilpotent.)

Let $R$ be a commutative ring, and let $I \subseteq R$ be an ideal. Prove that the following two statements are equivalent.

(a) The quotient ring $R/I$ contains no nilpotents.

(b) For every element $b \in R$ such that $b^m \in I$ for some positive integer $m \geq 1$, we have $b \in I$.

**Solution:** (a) $\Rightarrow$ (b): Given arbitrary $b \in R$, suppose $b^m \in I$ for some integer $m \geq 1$. Then

$$(I + b)^m = I + b^m = I + 0,$$

where the second equality is by the coset criterion. Since $R/I$ contains no nilpotents, we have $I + b = I + 0$. Thus, $b \in I$ by the coset criterion.

(b) $\Rightarrow$ (a): Suppose $R/I$ contains an element $I + b$ such that $(I + b)^n = I + 0$ for some integer $n \geq 1$; we need to show that $I + b$ is already the zero element $I + 0$. We have

$$I + b^n = (I + b)^n = I + 0,$$

and hence $b^n \in I$ by the coset criterion. By assumption (b), we have $b \in I$, and therefore $I + b = I + 0$ by the coset criterion, as desired. QED