Solutions to the Analysis problems on the Comprehensive Examination of January 31, 2014

1. (6 points)

   (a) State the Bolzano-Weierstrass Theorem for sequences of real numbers.

   Solution: Every bounded sequence of real numbers contains a convergent subsequence.

   (b) Give an example of a sequence that does not have a convergent subsequence.

   Solution: There are many solutions to this problem. Here is one: $\{1, 2, 3, 4, 5, \ldots\}$.

2. (4 points) Find all values of x for which the following series converges. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n2^n}$$

Solution: Recall the Ratio Test, which states that $\sum a_n$ converges if $\{a_n\}$ is a sequence of nonzero terms satisfying $\lim |a_{n+1}/a_n| < 1$, and diverges if $\lim |a_{n+1}/a_n| > 1$. Applying the Test to the above series, we have

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-1)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(-1)^n (x-1)^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{n(x-1)}{2(n+1)} \right| \\
&= \lim_{n\to\infty} \left| \frac{n}{n+1} \right| \left| \frac{x-1}{2} \right| \\
&= \frac{|x-1|}{2}.
\end{aligned}$$

So the series converges when

$$\frac{|x-1|}{2} < 1 \iff |x-1| < 2 \iff -1 < x < 3,$$

and diverges when $\dfrac{|x-1|}{2} > 1$, which means $x < -1$ or $x > 3$.

Lastly we check convergence on the endpoints. At $x = -1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n (-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges as it is a p-series with $p = 1$; at $x = 3$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which is an alternating series with the absolute values of its terms strictly decreasing and converging to zero, so it converges by the Alternating Series Test.

To summarize, the series converges for $x \in (-1, 3]$.

3. (10 points) Use induction to prove that

$$n! > 2^n$$

for all positive integers n greater than or equal to 4.

Solution: *Base case:* $n = 4$. Then $n! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ and $2^n = 2^4 = 16$, and since $24 > 16$ the statement is valid.

Induction step: Suppose the statement is true for positive integer $n \geq 4$. Then $n! > 2^n$. If we can show it is true for $n + 1$ as well then the proof is complete. Since $n \geq 4$, we may multiply across the inequality by $n + 1$ to get

$$(n+1) \cdot n! > 2^n(n+1).$$

But $n \geq 4$ implies $n + 1 \geq 2$, so we obtain

$$(n+1) \cdot n! > 2^n(n+1) > 2^n \cdot 2,$$

which implies

$$(n+1)! > 2^{n+1},$$

as desired.

4. (10 points)

(a) Let f be a real-valued function defined on $\mathbb{R}$. State the ε-δ definition of what it means for f to be continuous at a point c.

Solution: f is continuous at c if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for every $x \in \mathbb{R}$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at c and that $\{x_n\}$ is a sequence of real numbers that converges to c. Prove that the sequence $\{f(x_n)\}$ converges to $f(c)$.

Solution: Suppose $\varepsilon > 0$ is arbitrary. Because f is continuous at c, we may fix a $\delta > 0$ such that, for every $y \in \mathbb{R}$, if $|y - c| < \delta$, then $|f(y) - f(c)| < \varepsilon$. Furthermore, because $\{x_n\} \to c$, we may fix an $N \in \mathbb{N}$ such that if $n \geq N$, then $|x_n - c| < \delta$.

Pick an arbitrary $n \geq N$. Then we have $|x_n - c| < \delta$, which means $|f(x_n) - f(c)| < \varepsilon$, which proves that $\{f(x_n)\} \to f(c)$.

5. (10 points) Let $f_n(x) = \frac{nx}{1+n^2x^2}$ for $n \in \mathbb{N}$.

(a) State the function f to which the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise.

Solution: (*Note:* As it is not specified in the problem, we assume the domain of f_n to be $\mathbb{R}$ for all n.) Note that when $x = 0$, $f_n(x) = 0\ \forall\ n \in \mathbb{N}$. Now assume $x \neq 0$. Then we have

$$f_n(x) = \frac{nx}{1+n^2x^2} = \frac{x}{1/n + nx^2}.$$

Since $x \neq 0$ is fixed and n approaches ∞, the denominator $1/n + nx^2$ approaches $+\infty$. Since the numerator is constant, we see that $f_n(x)$ tends to zero as n approaches ∞. Thus $\{f_n(x)\} \to f(x) = 0\ \forall\ x \in \mathbb{R}$.

(b) Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[1, \infty)$.

Solution: Given an arbitrary $\varepsilon > 0$, fix $N \in \mathbb{N}$ such that $N > 1/\varepsilon$. Take an arbitrary $x \in [1, \infty)$ and $n \geq N$. Then

$$|f_n(x) - f(x)| = \left| \frac{nx}{1+n^2x^2} - 0 \right| = \frac{nx}{1+n^2x^2} < \frac{nx}{n^2x^2} = \frac{1}{nx} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

where $\dfrac{1}{nx} \leq \dfrac{1}{n}$ follows from $x \geq 1$. Thus $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and all $x \in [1, \infty)$. It follows that $\{f_n\} \to f$ uniformly on $[1, \infty)$.