Solutions to the Multivariable Calculus and Linear Algebra problems on the Comprehensive Examination of January 31, 2014

There are 9 problems (10 points each, totaling 90 points) on this portion of the examination. Show all of your work.

1. Find the critical points of the function \( f(x, y) = x^4 - 4xy + 2y^2 \) and classify as a local maximum, local minimum, or a saddle point.

Solution: Since \( f \) is a polynomial, it is differentiable on \( \mathbb{R}^2 \). The critical points occur when

\[
\begin{align*}
    f_x(x, y) &= 4x^3 - 4y = 0 \\
    f_y(x, y) &= -4x + 4y = 0
\end{align*}
\]

The second equation gives \( x = y \), and substituting it into the first gives \( 4x^3 - 4x = 0 \), or \( x(x + 1)(x - 1) = 0 \). Thus \( x = 0 \) or \( x = \pm 1 \). Therefore the critical points of \( f \) are \((0, 0)\), \((1, 1)\), and \((-1, -1)\).

To classify of the critical points, we use the second derivative test. First let us compute the second derivatives:

\[
\begin{align*}
    f_{xx}(x, y) &= 12x^2 \\
    f_{xy}(x, y) &= -4 \\
    f_{yy}(x, y) &= 4 \\
    D(x, y) &= f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2 = 48x^2 - 16
\end{align*}
\]

\( D(0, 0) = -16 < 0 \), so \((0, 0)\) is a saddle point; \( D(1, 1) = 32 > 0 \) and \( f_{xx}(1, 1) = 12 > 0 \), so \((1, 1)\) is a local minimum; and \( D(-1, -1) = 32 > 0 \), \( f_{xx}(-1, -1) = 12 > 0 \), so \((-1, -1)\) is also a local minimum.

2. Suppose the plane \( z = 2x - y - 1 \) is tangent to the graph of \( z = f(x, y) \) at \( P = (5, 3) \).

(a) Determine \( f(5, 3) \), \( \frac{\partial f}{\partial x}(5, 3) \) and \( \frac{\partial f}{\partial y}(5, 3) \).

Solution: We know the graphs of \( z = 2x - y - 1 \) and \( z = f(x, y) \) intersect at \( P(5, 3, f(5, 3)) \), so \( f(5, 3) = 2(5) - 3 - 1 = 6 \). Furthermore, recall that an equation of the tangent plane to \( z = f(x, y) \) at the point \( P(x_0, y_0, z_0) \) is

\[
z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)
\]

So we have

\[
z = 2x - y - 1 = 6 + \frac{\partial f}{\partial x}(5, 3)(x - 5) + \frac{\partial f}{\partial y}(5, 3)(y - 3).
\]

Comparing coefficients of \( x \) and \( y \), we obtain

\[
\frac{\partial f}{\partial x}(5, 3) = 2 \quad \frac{\partial f}{\partial y}(5, 3) = -1.
\]

(b) Estimate \( f(5.2, 2.9) \).

Solution: We use the linear approximation of \( f \) at \((5, 3)\):

\[
f(5.2, 2.9) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)
\]
\[
\approx f(5, 3) + \frac{\partial f}{\partial x}(5, 3)(5.2 - 5) + \frac{\partial f}{\partial y}(5, 3)(2.9 - 3)
\approx 6 + 2 \cdot .2 + (-1) \cdot (-1) = 6.5.
\]

Here is another way to do this. Near \((5, 3)\), the graph \(z = f(x, y)\) is approximated by the tangent plane at \((5, 3)\), which is given as \(z = 2x - y - 1\). Thus
\[
f(5.2, 2.9) \approx 2(5.2) - (2.9) - 1 = 6.5.
\]

3. Calculate the volume of the region inside sphere \(x^2 + y^2 + z^2 = a^2\) and outside the cylinder \(x^2 + y^2 = b^2\), where \(a > b\), by using an appropriate double integral.

Solution: We are removing a vertical cylinder of radius \(b\) from a sphere of radius \(a\). When we think of this as double integral, the region in the plane is
\[
R = \{(x, y) \mid b \leq x^2 + y^2 \leq a\} = \{(r, \theta) \mid 0 \leq \theta \leq 2\pi, b \leq r \leq a\}.
\]
The “top” of the figure is the top half of the sphere, given by \(z = \sqrt{a^2 - r^2}\), and the “bottom” is the bottom half of the sphere, given by \(z = -\sqrt{a^2 - r^2}\). Here, we are using cylindrical coordinates. Then the double integral giving the volume is
\[
V = \int \int_R \sqrt{a^2 - r^2} - (-\sqrt{a^2 - r^2}) dA
\]
\[
= \int_0^{2\pi} \int_b^a 2r \sqrt{a^2 - r^2} dr d\theta
\]
\[
= 2\pi \left[ -\frac{2}{3} (a^2 - r^2)^{3/2} \right]_b^a
\]
\[
= -\frac{4\pi}{3} \left( (a^2 - a^2)^{3/2} - (a^2 - b^2)^{3/2} \right)
\]
\[
= \frac{4\pi}{3} (a^2 - b^2)^{3/2}.
\]

4. Suppose that \(r(t) = (3\sqrt{2}t, e^{-3t}, e^{3t})\) describes the position of an object at time \(t\).

(a) Calculate the acceleration of the object at time \(t\).

Solution: The velocity and acceleration of the object at time \(t\) are
\[
v(t) = r'(t) = (3\sqrt{2}, -3e^{-3t}, 3e^{3t})
\]
\[
a(t) = r''(t) = (0, 9e^{-3t}, 9e^{3t}).
\]

(b) Calculate the speed of the object at time \(t\). Simplify by factoring the expression under the square root.

Solution: The speed of the object at time \(t\) is
\[
|v(t)| = \sqrt{(3\sqrt{2})^2 + (-3e^{-3t})^2 + (3e^{3t})^2}
\]
\[
= \sqrt{18 + 9e^{-6t} + 9e^{6t}} = \sqrt{9(e^{6t} + 2 + e^{-6t})}
\]
\[
= 3\sqrt{(e^{3t} + e^{-3t})^2} = 3(e^{3t} + e^{-3t}).
\]
(c) Calculate the distance traveled by the object between times \( t = 0 \) and \( f = 1 \).

**Solution:** The total distance traveled by the object between \( t = 0 \) and \( t = 1 \) is

\[
D = \int_0^1 |v(t)| \, dt = \int_0^1 3e^{3t} + 3e^{-3t} \, dt = e^3 - e^{-3t}
\]

5. Consider the function

\[
f(x, y) = \begin{cases} 
3y^3 & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}
\]

(a) Show that \( f \) is continuous at \((0, 0)\).

**Solution:** Recall that \( f \) is continuous at \((0, 0)\) iff \( \lim_{(x,y) \to (0,0)} f(x, y) = f(0, 0) \).

Hence we must show that \( \lim_{(x,y) \to (0,0)} \frac{3y^3}{x^2 + y^2} = 0 \). In polar coordinates, \( x = r \cos \theta, y = r \sin \theta \), and \((x,y) \to 0\) becomes \( r \to 0 \). Then the limit is

\[
\lim_{(x,y) \to (0,0)} \frac{3y^3}{x^2 + y^2} = \lim_{r \to 0} \frac{3r^3 \sin^3 \theta}{r^2} = \lim_{r \to 0} 3r \sin^3 \theta = 0,
\]

where the last equality follows since \( 3r \to 0 \) and \( \sin^3 \theta \) is bounded.

(b) Find \( f_x(0, 0) \) and \( f_y(0, 0) \).

**Solution:** We will directly apply the definition of partial derivatives.

\[
f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0
\]

and

\[
f_y(0, 0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{3h^3 - 0}{h} = \lim_{h \to 0} 3h = 3.
\]

6. Suppose \( T : V \to V \) is a linear transformation, \( B = \{b_1, b_2, b_3\} \) is a basis for \( V \), and the matrix representation of \( T \) with respect to \( B \) is

\[
\begin{bmatrix}
2 & 3 & 5 \\
7 & 11 & -3 \\
-1 & 19 & 0
\end{bmatrix}.
\]

Determine \( T(2b_1 + 4b_3) \) as a linear combination of \( b_1, b_2, \) and \( b_3 \).

**Solution:** The coordinate vector of \( 2b_1 + 4b_3 \) relative to \( B \) is \( a = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \). Then the coordinate vector of \( T(2b_1 + 4b_3) \) relative to \( B \) is

\[
b = Aa = \begin{bmatrix}
2 & 3 & 5 \\
7 & 11 & -3 \\
-1 & 19 & 0
\end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \\ -2 \end{bmatrix}.
\]

Hence

\[
T(2b_1 + 4b_3) = 24b_1 + 2b_2 - 2b_3.
\]
7. Let \( A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 2 \\ 3 & 0 & 1 \end{bmatrix} \).

(a) Compute the eigenvalue(s) of \( A \).

**Solution:** The eigenvalues of \( A \) are the roots of its characteristic polynomial

\[
\det(A - \lambda I) = \det \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 2 \\ 3 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 2 - \lambda & 0 & 0 \\ 2 & 6 - \lambda & 2 \\ 3 & 0 & 1 - \lambda \end{bmatrix}
\]

Thus \( A \) has three eigenvalues: \( \lambda = 1 \), \( \lambda = 2 \), and \( \lambda = 6 \).

(b) Find an invertible matrix \( C \) such that \( C^{-1}AC \) is diagonal.

**Solution:** We must first compute bases of the eigenspaces corresponding to the eigenvalues of \( A \).

\( \lambda = 1 \): The eigenspace is the solution space of \((A - I) \bar{x} = \vec{0}\). Being homogeneous, we need only row reduce the coefficient matrix:

\[
\begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 2 \\ 3 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2/5 \\ 0 & 0 & 0 \end{bmatrix},
\]

so we have \( x_1 = x_2 + (2/5)x_3 = 0 \) with free variable \( x_3 \). The solutions are

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ (-2/5)x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -2/5 \\ 1 \end{bmatrix}.
\]

Thus the eigenspace of \( A \) associated to \( \lambda = 1 \) has basis \( \left\{ \begin{bmatrix} 0 \\ -2/5 \\ 1 \end{bmatrix} \right\} \).

\( \lambda = 2 \): The eigenspace is the solution space of \((A - 2I) \bar{x} = \vec{0}\). Being homogeneous, we row reduce the coefficient matrix:

\[
\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 2 \\ 3 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{bmatrix},
\]

so we have \( x_1 - (1/3)x_3 = x_2 + (2/3)x_3 = 0 \) with free variable \( x_3 \). The solutions are

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/3)x_3 \\ (2/3)x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \end{bmatrix}.
\]
Thus the eigenspace of $A$ associated to $\lambda = 2$ has basis \[
\left\{ \begin{bmatrix}
\frac{1}{3} \\
-\frac{2}{3} \\
1
\end{bmatrix} \right\}.
\]

$\lambda = 6$: The eigenspace is the solution space of $(A - 6I) \vec{x} = \vec{0}$. As above, we row reduce the coefficient matrix:

\[
\begin{bmatrix}
-4 & 0 & 0 \\
2 & 0 & 2 \\
3 & 0 & -5
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix},
\]

so we have $x_1 = x_3 = 0$ with free variable $x_2$. The solutions are

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\]

Thus the eigenspace of $A$ associated to $\lambda = 6$ has basis \[
\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.
\]

Finally, because $A$ has three linearly independent eigenvectors, it is diagonalizable (the problem implies this anyway). Its eigenvectors form the columns of the desired matrix

\[
C = \begin{bmatrix}
0 & \frac{1}{3} & 0 \\
-\frac{2}{5} & -\frac{2}{3} & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]

(\text{Note: One can check that } C^{-1}AC \text{ is diagonal, but doing so is not recommended because the calculations are extremely time-consuming.})

8. Let $A = \begin{bmatrix}
1 & -1 & 1 & 3 \\
-1 & 1 & 0 & 2 \\
2 & -2 & 4 & \alpha
\end{bmatrix}$, where $\alpha$ is a real number.

(a) For what values of $\alpha$ does $Ax = b$ have at least one solution for all $b \in \mathbb{R}^3$?

\textbf{Solution:} Let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. We can row reduce to obtain

\[
\begin{bmatrix}
1 & -1 & 1 & 3 & b_1 \\
-1 & 1 & 0 & 2 & b_2 \\
2 & -2 & 4 & \alpha & b_3
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1 & 1 & 3 & b_1 \\
0 & 0 & 1 & b_1 + b_2 \\
0 & 0 & 2 & \alpha - 6 & b_3 - 2b_1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1 & 0 & 2 & -b_2 \\
0 & 0 & 1 & 1 & b_1 + b_2 \\
0 & 0 & 0 & \alpha - 16 & b_3 - 4b_1 - 2b_2
\end{bmatrix}
\]

To finish the row reduction, there are two cases, depending on $\alpha - 8$:

- $\alpha - 16 \neq 0$: We can divide the last row by $\alpha - 16$ to obtain a leading 1 in the 4th column. Hence we have at least one solution, no matter how we choose $b \in \mathbb{R}^3$.

- $\alpha - 16 = 0$: If we pick $b \in \mathbb{R}^3$ so that $b_3 - 4b_1 - 2b_2 \neq 0$, then the last row will have a leading 1 in the last column of the augmented matrix, which implies that the system has no solution. So $b \in \mathbb{R}^3$ can be chosen so that there is no solution.

\textbf{Conclusion} $Ax = b$ has at least one solution for all $b \in \mathbb{R}^3$ if and only if $\alpha \neq 16$.  

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(b) For the remainder of the problem set $\alpha = 11$. Find the general solution of $Ax = 0$.

**Solution:** Set $\alpha = 11$ in the partial row reduction of part (a). If we focus on the coefficient matrix ($Ax = 0$ is homogeneous), we get the row reduction

$$
\begin{bmatrix}
1 & -1 & 1 & 3 \\
-1 & 1 & 0 & 2 \\
2 & -2 & 4 & 11
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & -1 & 0 & 2 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 11 - 8
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 11
\end{bmatrix},
$$

which gives the equations $x_1 - x_2 = x_3 = x_4 = 0$ with free variable $x_2$. Thus the general solution is

$$
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
x_2 \\
0 \\
0
\end{bmatrix} = x_2
\begin{bmatrix}
1 \\
1 \\
0 \\
0
\end{bmatrix}.
$$

9. Suppose $\{u, v\}$ is a basis for a vector space $V$. Prove that $\{u + 2v, 3u - v\}$ is also a basis for $V$.

**Solution:** Our hypothesis implies that $V$ has dimension 2, and we are asked to prove that $\{u + 2v, 3u - v\}$ is a basis for $V$. Recall that $n$ vectors in an $n$-dimensional vector basis form a basis $\iff$ they span $\iff$ they are linearly independent. Here, it is easier to show that $u + 2v, 3u - v$ are linearly independent. Suppose $a(u + 2v) + b(3u - v) = 0$, where $a, b \in \mathbb{R}$. Then

$$
au + 2av + 3bu - bv = 0
\Rightarrow (a + 3b)u + (2a - b)v = 0.
$$

Since $\{u, v\}$ is a basis for $V$, we know that $u$ and $v$ are linearly independent. Therefore the last equation implies that

$$
 a + 3b = 0 \quad \text{and} \quad 2a - b = 0.
$$

It is easy to show that the only solution to the above system of equations is $a = b = 0$, which in turn implies that $u + 2v$ and $3u - v$ are linearly independent. As noted above, $V$ has dimension 2, so that any set of 2 linearly independent vectors in $V$ is a basis. Therefore, $\{u + 2v, 3u - v\}$ is a basis for $V$. 

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