Solutions to the Algebra problems on the Comprehensive Examination of January 30, 2015

1. **(25 points).** Let G and H be groups, let $B \subseteq H$ be a subgroup, and let $\phi : G \to H$ be a homomorphism. Define

$$A = \{g \in G \mid \phi(g) \in B\}.$$

(A *is called the inverse image of* B *under* ϕ.) Prove that A is a subgroup of G.

[Note: this is a standard theorem in Math 350. You are being asked to prove this theorem, so obviously you cannot simply quote that theorem.]

Solution: We must show that A is (1) nonempty, (2) closed under multiplication, and (3) closed under inverses.

(1) Let e_G be the identity of G and e_H the identity of H. We have $\phi(e_G) = e_H \in B$, since ϕ is a homomorphism and B is a subgroup. So $e_G \in A$, and hence $A \neq \varnothing$.

(2) Given arbitrary $x, y \in A$, We have

$$\phi(xy) = \phi(x)\phi(y) \in B,$$

since $\phi(x), \phi(y) \in B$, and B is closed under multiplication. So $xy \in A$.

(3) Given arbitrary $x \in A$, we have

$$\phi(x^{-1}) = \phi(x)^{-1} \in B,$$

since $\phi(x) \in B$, and B is closed under inverses. So $x^{-1} \in A$. QED

2. **(25 points).** Let G be an abelian group, and define

$$T = \{g \in G \mid g \text{ has finite order}\}.$$

It is a fact, which you may assume, that T is a normal subgroup of G.

Prove that the only element of the quotient group G/T that has finite order is the identity element.

Solution: Given an arbitrary element $Ta \in G/T$ of finite order $n \geq 1$, with $a \in G$, we have

$$Ta^n = (Ta)^n = Te,$$

since Te is the identity element of G/T. By the coset criterion, then, $a^n \in T$. By definition of T, then, a^n has finite order; let $m = o(a^n) \geq 1$. So

$$a^{nm} = (a^n)^m = e,$$

and hence $a \in G$ has finite order $nm \geq 1$. Therefore, $a \in T$. Thus, $Ta = Te$ by the coset criterion. QED

3. **(25 points).** Recall that S_n denotes the group of permutations of the set $\{1, 2, ..., n\}$.

(a) Let $\sigma = (1,4,3)(3,5)(2,7,5)(1,6,2,4,7) \in S_7$. Write σ as a product of disjoint cycles.

Solution: $\sigma = (1,6,7,4)(2,3,5)$.

(b) Determine whether σ is an even or an odd permutation.

Solution: Since σ is a product of a 4-cycle (odd, since 4 is even) and a 3-cycle (even, since 3 is odd). So σ is odd + even = odd.

(c) Give an example of an **even** permutation $\tau \in S_7$ of order 4. Don't forget to justify your answer.

Solution: There are many acceptable solutions to this problem. Here is one: $\tau = (1,2)(3,4,5,6)$. Its order is $o(\tau) = \text{lcm}(2,4) = 4$, and as a product of a 2-cycle (odd) and a 4-cycle (odd), it is odd + odd = even.

4. **(25 points).** Let R be a ring.

(a) Define what it means for a subset $I \subseteq R$ to be an **ideal** of R. If you use any other technical terms like "closed," "subring," "subgroup," etc., you must fully define those terms as well.

Solution: $I \subseteq R$ is said to be an ideal of R if

i. I is nonempty;

ii. for every x, $y \in I$, we have $x - y \in I$; and

iii. for every $a \in R$ and $b \in I$, we have ab, $ba \in I$.

(b) Assume R is commutative, and let $I, J \subseteq R$ be ideals of R. Define

$$IJ = \{x_1y_1 + \cdots + x_ny_n \mid n \geq 1 \text{ and } x_i \in I,\ y_i \in J\}.$$

That is, IJ is the set of all finite sums of products of an element of I times an element of J. Prove that IJ is an ideal of R.

Solution: We check each of the three criteria listed above.

i. There exist $x \in I$ and $y \in J$, since $I, J \neq \varnothing$. So $xy \in IJ$, and so $IJ \neq \varnothing$.

ii. Given $a, b \in IJ$, write

$$a = x_1y_1 + \cdots + x_my_m \quad \text{and} \quad b = x_1'y_1' + \cdots + x_n'y_n',$$

where $m, n \geq 1$ and $x_i, x_i' \in I$, $y_i, y_i' \in J$. Then

$$\begin{aligned} a - b &= x_1y_1 + \cdots + x_my_m - (x_1'y_1' + \cdots + x_n'y_n') \\ &= x_1y_1 + \cdots + x_my_m + (-x_1')y_1' + \cdots + (-x_n')y_n' \in IJ, \end{aligned}$$

since each $-x_i' \in I$ because I is an ideal, and therefore $a - b$ is a sum of $m + n \geq 1$ products xy with $x \in I$ and $y \in J$.

iii. Given $r \in R$ and $a \in IJ$, write $a = x_1y_1 + \cdots + x_ny_n$, where $n \geq 1$ and $x_i \in I$, $y_i \in J$. Then

$$ra = r(x_1y_1 + \cdots + x_ny_n) = (rx_1)y_1 + \cdots + (rx_n)y_n \in IJ,$$

since each $rx_i \in I$ because I is an ideal. Since R is commutative, we also have $ar = ra \in IJ$. QED