Solutions to the Analysis problems on the Comprehensive Examination of January 30, 2015

1. (a) (2 points) Explain what it means to say that a sequence (x_n) of real numbers is *monotone*.

   Solution: (x_n) is monotone if either $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ or $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$.

   (b) (2 points) Give a precise statement of the Monotone Convergence Theorem.

   Solution: If a sequence of real numbers is monotone and bounded, then it converges.

   (c) (2 points) Give an example of a monotone sequence of real numbers that does not converge.

   Solution: There are many solutions to this problem. Here is one: $(1, 2, 3, 4, 5, \ldots)$.

2. (a) (4 points) State the ϵ-δ definition of what it means for the function f to be continuous at c.

   Solution: f is continuous at c if, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ (where A is the domain of f) and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

   (b) (6 points) Suppose that f and g are functions that are continuous at c. Prove that the function $(f + g)$, given by

$$(f + g)(x) = f(x) + g(x),$$

   is also continuous at c.

   Solution: (*Caution!* Because it is not specified what the domains of f and g are, we may not assume they are the same.) Let A be the domain of f and B the domain of g. Then the domain of $(f + g)$ is $A \cap B$.

   Suppose $\epsilon > 0$ is arbitrary. Because f is continuous at c, there is $\delta_1 > 0$ such that if $x \in A$ and $|x - c| < \delta_1$, then $|f(x) - f(c)| < \epsilon/2$. Similarly, because g is continuous at c, there is $\delta_2 > 0$ such that if $x \in B$ and $|x - c| < \delta_2$, then $|g(x) - g(c)| < \epsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\}$. If $x \in A \cap B$ and $|x - c| < \delta$, then

$$\begin{aligned} |(f+g)(x) - (f+g)(c)| &= |(f(x) + g(x)) - (f(c) - g(c))| \\ &= |(f(x) - f(c)) + (g(x) - g(c))| \\ &\leq |f(x) - f(c)| + |g(x) - g(c)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

   This proves that $(f + g)$ is continuous at c.

3. (a) (6 points) Show that the sequence of functions (x^n) converges pointwise, as $n \to \infty$, for x in the interval $[0, 1]$.

Solution: Note that for $x = 1$, (x^n) is constant and converges to 1, whereas for $x \in [0, 1)$, $x^n \to 0$ as $n \to \infty$. Hence

$$(x^n) \to f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1. \end{cases}$$

(*Note:* It is also possible to show (x^n) converges pointwise without explicitly calculating the limit; it suffices to show that for all $x \in [0, 1]$, the sequence of real numbers (x^n) converges. Note that for $x \in [0, 1]$, $0 \leq x^{n+1} \leq x^n \ \forall \ n \in \mathbb{N}$, so by the Monotone Convergence Theorem, (x^n) converges. However, you will still need to find the limit function in part (c).)

(b) (4 points) Explain precisely what it means to say that a sequence (f_n) of functions $f_n : [0, 1] \to \mathbb{R}$ converges uniformly to a function f.

Solution: (f_n) converges uniformly on $[0, 1]$ to f if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $x \in [0, 1]$ and all $n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$.

(c) (4 points) Does the sequence (x_n) converge uniformly on $[0, 1]$? Explain your answer.

Solution: No, (x^n) does not converge uniformly on $[0, 1]$. We know that for each $n \in \mathbb{N}$, x^n is continuous on $[0, 1]$, so if (x^n) were uniformly convergent, its limit would also be continuous on $[0, 1]$ (by a basic theorem on uniform convergence). However, we know from part (a) that the limit function f is not continuous at $x = 1$.

(*Note:* This problem can also be done by directly applying the definition of uniform convergence, but since you are allowed to use known theorems (provided you state them clearly and explain how they apply), the above method is faster.)

4. (a) (3 points) State the Mean Value Theorem as it applies to a function f. (Be sure to include all the necessary hypotheses on the function f.)

Solution: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) (7 points) Suppose a function f has the following properties:

- the domain of f is $\mathbb{R}$;
- f is differentiable at every real number;
- $f'(x) = 0$ for every real number x.

Prove that there is a constant K such that $f(x) = K$ for every real number x. (Hint: show that $f(x) = f(0)$ for all x.)

Solution: Fix an arbitrary $x \in \mathbb{R} \setminus \{0\}$. First assume $x > 0$. Then f is differentiable on $(0, x)$ and, because it is differentiable on $\mathbb{R}$, continuous on $[0, x]$. By the Mean Value Theorem, we may choose a $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}.$$

But, by assumption, $f'(c) = 0$, and since $x \neq 0$, it must be true that $f(x) - f(0) = 0$. Next assume $x < 0$. Repeating the above argument with $[x, 0]$ in place of $[0, x]$ gives $f(0) - f(x) = 0$. Finally, $f(x) - f(0) = 0$ is obvious when $x = 0$. Hence we have

$$f(x) = f(0) \ \forall \ x \in \mathbb{R}$$

and $f(0)$ is the desired constant K.