Solutions to the Multivariable Calculus and Linear Algebra problems on the Comprehensive Examination of January 30, 2015

There are 9 problems (10 points each, totaling 90 points) on this portion of the examination. Show all of your work.

1. Find the critical points of the function \( f(x, y) = x^3 + y^3 - 3xy \) and classify each as a local maximum, local minimum, or a saddle point.

**Solution:** Since \( f \) is a polynomial, we know it is differentiable on \( \mathbb{R}^2 \). The critical points of \( f \) occur when

\[
\begin{align*}
    f_x(x, y) &= 3x^2 - 3y = 0 \quad \text{and} \quad f_y(x, y) = 3y^2 - 3x = 0.
\end{align*}
\]

Then we have \( y = x^2 \), and the second equation becomes \( 3(x^2)^2 - 3x = 0 \), or \( x(x^3 - 1) = 0 \). Then the real roots are \( x = 0 \) or \( x = 1 \), with the corresponding \( y = 0 \) and 1, respectively. Therefore the critical points of \( f \) are \((0, 0)\) and \((1, 1)\).

Now we calculate the second derivatives:

\[
\begin{align*}
    f_{xx} &= 6x \\
    f_{xy} &= -3 \\
    f_{yy} &= 6y \\
    D &= f_{xx}f_{yy} - f_{xy}^2 = (6x)(6y) - (-3)^2 = 36xy - 9.
\end{align*}
\]

Applying the second derivative test, we see that \( D(0, 0) = -9 < 0 \), so \((0, 0)\) is a saddle point; \( D(1, 1) = 36 - 9 > 0 \) and \( f_{xx}(1, 1) = 6 > 0 \), so \((1, 1)\) is a local minimum.

2. Let \( C \) be the triangle with vertices \((0, 0)\), \((1, 1)\), and \((0, 1)\), oriented counterclockwise, and let \( \mathbf{F}(x, y) = \langle xy, x^2 \rangle \).

   (a) According to Green’s theorem, the line integral

   \[
   \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C xy \, dx + x^2 \, dy
   \]

   is equal to a certain double integral. Set up this double integral.

   **Solution:** Let \( R \) be the region bound by \( C \). Then

   \[
   \int_C xy \, dx + x^2 \, dy = \iint_R \frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y} xy \, dA = \iint_R 2x - x \, dA = \iint_R x \, dA.
   \]

   (b) Verify Green’s Theorem in this case by evaluating both the line integral and the double integral of part (a).

   **Solution:** Line integral: First it is necessary to parametrize the three segments of \( C \).

   \[
   \begin{align*}
   C_1 : x &= t, \ y = t \Rightarrow dx = dt, \ dy = dt, \ 0 \leq t \leq 1 \\
   C_2 : x &= 1 - t, \ y = 1 \Rightarrow dx = -dt, \ dy = 0 \, dt, \ 0 \leq t \leq 1 \\
   C_3 : x &= 0, \ y = 1 - t \Rightarrow dx = 0 \, dt, \ dy = -dt, \ 0 \leq t \leq 1.
   \end{align*}
   \]
Thus

\[
\int_C xy \, dx + x^2 \, dy = \int_{C_1} xy \, dx + x^2 \, dy + \int_{C_2} xy \, dx + x^2 \, dy + \int_{C_2} xy \, dx + x^2 \, dy \\
= \int_0^1 (t^2 + t^2) \, dt + \int_0^1 ((1 - t)\langle -1 \rangle + (1 - t)\langle 0 \rangle) \, dt \\
\quad + \int_0^1 (0(1 - t)\langle 0 \rangle + 0^2\langle -1 \rangle) \, dt \\
= \int_0^1 2t^2 + t - 1 \, dt = \frac{2t^3}{3} + \frac{t^2}{2} - t \bigg|_0^1 = \frac{1}{6}.
\]

Double integral: We may express \( R = \{(x, y) \mid 0 \leq x \leq y, \ 0 \leq y \leq 1\} \). Then

\[
\iint_R x \, dA = \int_0^1 \int_x^1 x \, dy \, dx = \int_0^1 x - x^2 \, dx = \frac{x^2}{2} - \frac{x^3}{3} \bigg|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.
\]

3. Let 

\[
f(x, y) = \begin{cases} 
\frac{x^4 + y^3 + xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}
\]

(a) Is \( f \) continuous at \((0, 0)\)? Justify your answer.

Solution: Recall that \( f \) is continuous at \((0, 0)\) iff \( \lim_{(x, y) \to (0, 0)} f(x, y) = f(0, 0) \).

Let \((x, y) \to (0, 0)\) along \( y = mx \), where \( m \) is any real number. Then

\[
f(x, y) = f(x, mx) = \frac{x^4 + (mx)^3 + x(mx)}{x^2 + (mx)^2} = \frac{x^2(2x^2 + m^3x + m)}{x^2(1 + m^2)} = \frac{x^2 + m^3x + m}{m^2 + 1}.
\]

Thus \( f(x, y) \to \frac{m}{m^2 + 1} \) as \((x, y) \to (0, 0)\) along \( y = mx \), so \( f \) has different limits along different paths. Therefore \( \lim_{(x, y) \to (0, 0)} f(x, y) \) does not exist, and \( f \) is not continuous at \((0, 0)\).

(b) Find \( f_x(0, 0) \) and \( f_y(0, 0) \).

Solution: We shall apply the definition of partial derivatives.

\[
f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h^4/h^2 - 0}{h} = \lim_{h \to 0} h = 0
\]

and

\[
f_y(0, 0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h^3/h^2 - 0}{h} = \lim_{h \to 0} 1 = 1.
\]

4. Find the directional derivative of the function \( f(x, y, z) = x\sqrt{yz + 1} \) at the point \((2, 1, 3)\) in the direction of the vector \( \langle 2, -1, 2 \rangle \).

Solution: The unit vector in the direction of \( \langle 2, -1, 2 \rangle \) is

\[
u = \frac{\langle 2, -1, 2 \rangle}{\|\langle 2, -1, 2 \rangle\|} = \frac{\langle 2, -1, 2 \rangle}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{\langle 2, -1, 2 \rangle}{3} = \langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \rangle.
\]
The directional derivative of $f$ in the direction of $u$ is therefore

$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot u = \left\langle \frac{xz}{\sqrt{yz + 1}}, \frac{xy}{2\sqrt{yz + 1}}, \frac{xy}{3\sqrt{yz + 1}} \right\rangle \cdot \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle$$

Thus

$$D_u f(2, 1, 3) = \frac{2}{3} \sqrt{(1)(3) + 1} - \frac{(2)(3)}{6\sqrt{(1)(3) + 1}} + \frac{(2)(1)}{3\sqrt{(1)(3) + 1}} = \frac{4}{3} - \frac{1}{2} + \frac{1}{3} = \frac{7}{6}.$$

5. Find the volume of the region that is inside both the sphere $x^2 + y^2 + z^2 = 25$ and the cylinder $x^2 + y^2 = 9$.

**Solution:** Let $E$ denote the region in question. In cylindrical coordinates, the cylinder is $r = 3$, and the sphere is $r^2 + z^2 = 25$. Then $E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, \, 0 \leq r \leq 3, \, -\sqrt{25 - r^2} \leq z \leq \sqrt{25 - r^2}\}$. The volume of $E$ is therefore

$$\int \int \int_E dV = \int_0^{2\pi} \int_0^3 \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} rz \, dr \, dz \, d\theta = \int_0^{2\pi} \int_0^3 2r \sqrt{25 - r^2} \, dz \, dr \, d\theta = 2\pi \left[ -\frac{2}{3} (25 - r^2)^{3/2} \right]_0^3 = -\frac{4\pi}{3} \left( 16^{3/2} - 25^{3/2} \right) = -\frac{4\pi}{3} \left( 64 - 125 \right) = \frac{244\pi}{3},$$

since $16^{3/2} = (4^2)^{3/2} = 4^3 = 64$ and $25^{3/2} = (5^2)^{3/2} = 5^3 = 125$.

6. Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear transformation $T(x) = Ax$, where

$$A = \begin{bmatrix} 1 & -2 & 2 & 7 \\ 2 & -4 & 2 & 8 \\ 1 & -2 & -1 & -2 \end{bmatrix}.$$

(a) Find a basis for the kernel, or nullspace, of $T$.

**Solution:** The kernel consists of the solutions of the homogeneous system $Ax = 0$. We first row reduce the matrix $A$:

$$\begin{bmatrix} 1 & -2 & 2 & 7 \\ 2 & -4 & 2 & 8 \\ 1 & -2 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
so we have \( x_1 - 2x_2 + x_4 = x_3 + 3x_4 = 0 \) with free variables \( x_2 \) and \( x_4 \). The general solution is

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 
\end{bmatrix} = x_2 \begin{bmatrix} 2x_2 - x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ -3 \end{bmatrix}.
\]

Thus a basis of the kernel of \( T \) is given by

\[
\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}.
\]

(b) What is the dimension of the range of \( T \)?

**Solution:** From part (a) we know \( \dim \ker T = 2 \), and \( \dim \text{domain } T = \text{number of columns of } A = 4 \). So the rank/nullity theorem tells us \( \dim \text{range } T = 4 - 2 = 2 \).

7. Suppose that \( V \) is a vector space and \( \vec{u} \) and \( \vec{v} \) are vectors in \( V \). Show that

\[
\text{Span}(\{3\vec{u} + \vec{v}, \vec{u} - \vec{v}\}) = \text{Span}(\{\vec{u}, \vec{v}\}).
\]

**Solution:** Denote \( S_1 = \{3\vec{u} + \vec{v}, \vec{u} - \vec{v}\} \) and \( S_2 = \{\vec{u}, \vec{v}\} \). In order to show \( \text{Span}(S_1) = \text{Span}(S_2) \), it suffices to show that

(1) Every vector in \( S_1 \) can be written as a linear combination of the vectors in \( S_2 \), and

(2) Every vector in \( S_2 \) can be written as a linear combination of the vectors in \( S_1 \).

(1) is obvious. (2) is not hard to show, either. We first want to express \( \vec{u} \) as a linear combination \( u = a(3\vec{u} + \vec{v}) + b(\vec{u} - \vec{v}) \) for some \( a, b \). Since \( a(3\vec{u} + \vec{v}) + b(\vec{u} - \vec{v}) = (3a + b)\vec{u} + (a - b)\vec{v} \), this means picking \( a \) and \( b \) so that

\[
3a + b = 1 \quad a - b = 0.
\]

One sees easily that the solution is \( a = b = \frac{1}{4} \). Thus \( \vec{u} = \frac{1}{4}(3\vec{u} + \vec{v}) + \frac{1}{4}(\vec{u} - \vec{v}) \). A similar procedure shows that \( \vec{v} = \frac{1}{4}(3\vec{u} + \vec{v}) - \frac{1}{4}(\vec{u} - \vec{v}) \).

8. Let

\[
A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}.
\]

(a) Find all eigenvalues of \( A \).

**Solution:** The eigenvalues of \( A \) are the roots of its characteristic polynomial

\[
\det(A - \lambda I_3) = \det \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} 5 - \lambda & 0 & 0 \\ 0 & 1 - \lambda & 2 \\ 1 & 2 & 4 - \lambda \end{pmatrix}
\]

\[
= (5 - \lambda) \det \begin{pmatrix} 1 - \lambda & 2 \\ 0 & 4 - \lambda \end{pmatrix} = (5 - \lambda)((1 - \lambda)(4 - \lambda) - 4)
\]

\[
= (5 - \lambda)(4 - 5\lambda + \lambda^2 - 4) = -\lambda(\lambda - 5)^2.
\]

Thus \( A \) has two eigenvalues: \( \lambda = 0 \) and \( \lambda = 5 \).
(b) Find, if possible, an invertible matrix $P$ such that $P^{-1}AP$ is diagonal, or show that no such matrix exists.

**Solution:** The existence of $P$ is equivalent to saying that $3 \times 3$ matrix $A$ is diagonalizable, i.e., has three linearly independent eigenvectors. The theory of eigenspaces provides the tools needed to decide whether or not $A$ is diagonalizable:

- The dimension of each eigenspace is bounded by the multiplicity of the corresponding eigenvalue in the characteristic polynomial. Thus the eigenspace of $\lambda = 0$ has dimension $\leq 1$ and the eigenspace of $\lambda = 5$ has dimension $\leq 2$.
- $A$ is diagonalizable $\iff$ the dimensions of the eigenspaces add up to $3 \iff$ the dimension of each eigenspace equals the multiplicity of the eigenvalue in the characteristic polynomial.

We begin with $\lambda = 5$. The eigenspace is the solution space of $(A - 5I_3)x = 0$. Being homogeneous, we need only row reduce the coefficient matrix:

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & -4 & 2 \\
1 & 2 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -1/2 \\
0 & 0 & 0
\end{bmatrix},
\]

which has rank 2. By the rank-nullity theorem, the dimension of the nullspace (= the eigenspace of 5) is $3 - 2 = 1$. This is strictly smaller than 2, which is the multiplicity of 5 as a root of the characteristic polynomial. Hence $A$ is not diagonalizable, and no such matrix $P$ exists.

**Alternate Solution:** The existence of $P$ is equivalent to saying that $3 \times 3$ matrix $A$ is diagonalizable, i.e., has three linearly independent eigenvectors. We compute the dimensions of the eigenspaces.

$\lambda = 5$. The eigenspace is the solution space of $(A - 5I_3)x = 0$. Being homogeneous, we need only row reduce the coefficient matrix:

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & -4 & 2 \\
1 & 2 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -1/2 \\
0 & 0 & 0
\end{bmatrix},
\]

which has rank 2. By the rank-nullity theorem, the dimension of the nullspace (= the eigenspace of 5) is $3 - 2 = 1$. Hence there is only one linearly independent eigenvector when $\lambda = 5$.

$\lambda = 0$. The eigenspace is the solution space of $(A - 0I_3)x = Ax = 0$. Being homogeneous, we need only row reduce the coefficient matrix:

\[
\begin{bmatrix}
5 & 0 & 0 \\
0 & 1 & 2 \\
1 & 2 & 4
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix},
\]

which has rank 2. By the rank-nullity theorem, the dimension of the nullspace (= the eigenspace of 0) is $3 - 2 = 1$. Hence there is only one linearly independent eigenvector when $\lambda = 0$.

It follows that there are only two linearly independent eigenvectors. So $A$ is not diagonalizable, and no such matrix $P$ exists.
9. Let $V$ be the vector space of polynomials of degree at most 2, and let

$$\mathcal{B} = \{1, x + 1, x^2 + x + 1\},$$

which is a basis for $V$. Suppose that $T : V \to V$ is a linear transformation, and that the matrix of $T$ relative to $\mathcal{B}$ is

$$\begin{bmatrix}
1 & 2 & 3 \\
3 & 0 & -1 \\
2 & 5 & 1
\end{bmatrix}.$$

Find $T(3x^2 + x + 2)$.

**Solution:** We first compute the coordinate vector of $3x^2 + x + 2$ relative to $\mathcal{B}$. Since

$$3x^2 + x + 2 = 3(x^2 + x + 1) - 2x - 1 = 3(x^2 + x + 1) - 2(x + 1) + 1 = 1(1) - 2(x + 1) + 3(x^2 + x + 1),$$

the coordinate vector of $3x^2 + x + 2$ is

$$\begin{bmatrix}
1 \\
-2 \\
3
\end{bmatrix}.$$ The coordinate vector of $T(3x^2 + x + 2)$ relative to $\mathcal{B}$ is therefore

$$\begin{bmatrix}
1 & 2 & 3 \\
3 & 0 & -1 \\
2 & 5 & 1
\end{bmatrix} \begin{bmatrix}
1 \\
-2 \\
3
\end{bmatrix} = \begin{bmatrix}
6 \\
0 \\
-5
\end{bmatrix}.$$ 

So $T(3x^2 + x + 2) = 6(1) + 0(x + 1) + (-5)(x^2 + x + 1) = -5x^2 - 5x + 1$. 
