Solutions to the Multivariable Calculus and Linear Algebra problems on the Comprehensive Examination of January 30, 2015

There are 9 problems (10 points each, totaling 90 points) on this portion of the examination. **Show all of your work.**

1. Find the critical points of the function $f(x, y) = x^3 + y^3 - 3xy$ and classify each as a local maximum, local minimum, or a saddle point.

   Solution: Since f is a polynomial, we know it is differentiable on $\mathbb{R}^2$. The critical points of f occur when

   $$f_x(x, y) = 3x^2 - 3y = 0 \qquad \text{and} \qquad f_y(x, y) = 3y^2 - 3x = 0.$$

   Then we have $y = x^2$, and the second equation becomes $3\left(x^2\right)^2 - 3x = 0$, or $x(x^3 - 1) = 0$. Then the real roots are $x = 0$ or $x = 1$, with the corresponding $y = 0$ and 1, respectively. Therefore the critical points of f are $(0, 0)$ and $(1, 1)$.

   Now we calculate the second derivatives:

   $$\begin{aligned} &f_{xx} = 6x \qquad f_{xy} = -3 \qquad f_{yy} = 6y \\ &D = f_{xx} f_{yy} - f_{xy}^2 = (6x)(6y) - (-3)^2 = 36xy - 9. \end{aligned}$$

   Applying the second derivative test, we see that $D(0, 0) = -9 < 0$, so $(0, 0)$ is a saddle point; $D(1, 1) = 36 - 9 > 0$ and $f_{xx}(1, 1) = 6 > 0$, so $(1, 1)$ is a local minimum.

2. Let C be the triangle with vertices $(0, 0)$, $(1, 1)$, and $(0, 1)$, oriented counterclockwise, and let $\mathbf{F}(x, y) = \langle xy, x^2 \rangle$.

   (a) According to Green's theorem, the line integral

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C xy\, dx + x^2\, dy$$

   is equal to a certain double integral. Set up this double integral.

   Solution: Let R be the region bound by C. Then

   $$\int_C xy\, dx + x^2\, dy = \iint_R \frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y} xy\, dA = \iint_R 2x - x\, dA = \iint_R x\ dA.$$

   (b) Verify Green's Theorem in this case by evaluating both the line integral and the double integral of part (a).

   Solution: *Line integral:* First it is necessary to parametrize the three segments of C.

   $$\begin{aligned} &C_1 : x = t,\ y = t \Rightarrow dx = dt,\ dy = dt,\ 0 \le t \le 1 \\ &C_2 : x = 1 - t,\ y = 1 \Rightarrow dx = -dt,\ dy = 0\, dt,\ 0 \le t \le 1 \\ &C_3 : x = 0,\ y = 1 - t \Rightarrow dx = 0\, dt,\ dy = -dt,\ 0 \le t \le 1. \end{aligned}$$

Thus

$$\begin{aligned}\int_C xy\,dx + x^2\,dy &= \int_{C_1} xy\,dx + x^2\,dy + \int_{C_2} xy\,dx + x^2\,dy + \int_{C_2} xy\,dx + x^2\,dy \\ &= \int_0^1 \left(t^2 + t^2\right) dt + \int_0^1 \left((1-t)(-1) + (1-t)^2(0)\right) dt \\ &\qquad + \int_0^1 \left(0(1-t)(0) + 0^2(-1)\right) dt \\ &= \int_0^1 2t^2 + t - 1\,dt = \frac{2t^3}{3} + \frac{t^2}{2} - t\,\Big|_0^1 = \frac{1}{6}.\end{aligned}$$

Double integral: We may express $R = \{(x, y) \mid 0 \le x \le y,\ 0 \le y \le 1\}$. Then

$$\iint_R x\ dA = \int_0^1 \int_x^1 x\ dy\,dx = \int_0^1 x - x^2\ dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

3. Let

$$f(x, y) = \begin{cases} \dfrac{x^4 + y^3 + xy}{x^2 + y^2} & \text{if } (x, y) \ne (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Is f continuous at $(0, 0)$? Justify your answer.

Solution: Recall that f is continuous at $(0, 0)$ iff $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$. Let $(x, y) \to (0, 0)$ along $y = mx$, where m is any real number. Then

$$f(x, y) = f(x, mx) = \frac{x^4 + (mx)^3 + x\,(mx)}{x^2 + (mx)^2} = \frac{x^2\,(x^2 + m^3x + m)}{x^2\,(1 + m^2)} = \frac{x^2 + m^3x + m}{m^2 + 1}.$$

Thus $f(x, y) \to \frac{m}{m^2+1}$ as $(x, y) \to (0, 0)$ along $y = mx$, so f has different limits along different paths. Therefore $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, and f is not continuous at $(0, 0)$.

(b) Find $f_x(0, 0)$ and $f_y(0, 0)$.

Solution: We shall apply the definition of partial derivatives.

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{h^4/h^2 - 0}{h} = \lim_{h\to 0} h = 0$$

and

$$f_y(0, 0) = \lim_{h\to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{h^3/h^2 - 0}{h} = \lim_{h\to 0} 1 = 1.$$

4. Find the directional derivative of the function $f(x, y, z) = x\sqrt{yz + 1}$ at the point $(2, 1, 3)$ in the direction of the vector $\langle 2, -1, 2\rangle$.

Solution: The unit vector in the direction of $\langle 2, -1, 2\rangle$ is

$$\mathbf{u} = \frac{\langle 2, -1, 2\rangle}{|\langle 2, -1, 2\rangle|} = \frac{\langle 2, -1, 2\rangle}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{\langle 2, -1, 2\rangle}{3} = \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle.$$

The directional derivative of f in the direction of $\mathbf{u}$ is therefore

$$\begin{aligned} D_{\mathbf{u}} f(x,y,z) &= \nabla f(x,y,z) \cdot \mathbf{u} \\ &= \left\langle \sqrt{yz+1}, \frac{xz}{2\sqrt{yz+1}}, \frac{xy}{2\sqrt{yz+1}} \right\rangle \cdot \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle \\ &= \frac{2}{3}\sqrt{yz+1} - \frac{xz}{6\sqrt{yz+1}} + \frac{xy}{3\sqrt{yz+1}}. \end{aligned}$$

Thus

$$\begin{aligned} D_{\mathbf{u}} f(2,1,3) &= \frac{2}{3}\sqrt{(1)(3)+1} - \frac{(2)(3)}{6\sqrt{(1)(3)+1}} + \frac{(2)(1)}{3\sqrt{(1)(3)+1}} \\ &= \frac{4}{3} - \frac{1}{2} + \frac{1}{3} = \frac{7}{6}. \end{aligned}$$

5. Find the volume of the region that is inside both the sphere $x^2+y^2+z^2=25$ and the cylinder $x^2+y^2=9$.

Solution: Let E denote the region in question. In cylindrical coordinates, the cylinder is $r=3$, and the sphere is $r^2+z^2=25$. Then $E=\{(r,\theta,z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 3,\ -\sqrt{25-r^2} \le z \le \sqrt{25-r^2}\}$. The volume of E is therefore

$$\begin{aligned} \iiint_E dV &= \int_0^{2\pi} \int_0^3 \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi} \int_0^3 2r\sqrt{25-r^2}\,dz\,dr\,d\theta \\ &= 2\pi \left[-\frac{2}{3}\left(25-r^2\right)^{3/2} \right]_0^3 \\ &= -\frac{4\pi}{3}\left((25-3^2)^{3/2} - (25-0^2)^{3/2}\right) \\ &= -\frac{4\pi}{3}\left(16^{3/2} - 25^{3/2}\right) = -\frac{4\pi}{3}(64-125) = \frac{244\pi}{3}, \end{aligned}$$

since $16^{3/2}=(4^2)^{3/2}=4^3=64$ and $25^{3/2}=(5^2)^{3/2}=5^3=125$.

6. Let $T:\mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation $T(\mathbf{x})=A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & -2 & 2 & 7 \\ 2 & -4 & 2 & 8 \\ 1 & -2 & -1 & -2 \end{bmatrix}.$$

(a) Find a basis for the kernel, or nullspace, of T.

Solution: The kernel consists of the solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$. We first row reduce the matrix A:

$$\begin{bmatrix} 1 & -2 & 2 & 7 \\ 2 & -4 & 2 & 8 \\ 1 & -2 & -1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so we have $x_1 - 2x_2 + x_4 = x_3 + 3x_4 = 0$ with free variables x_2 and x_4. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

Thus a basis of the kernel of T is given by

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

(b) What is the dimension of the range of T?

Solution: From part (a) we know dim ker $T = 2$, and dim domain T = number of columns of $A = 4$. So the rank/nullity theorem tells us dim range $T = 4 - 2 = 2$.

7. Suppose that V is a vector space and $\vec{u}$ and $\vec{v}$ are vectors in V. Show that

$$\text{Span}(\{3\vec{u} + \vec{v}, \vec{u} - \vec{v}\}) = \text{Span}(\{\vec{u}, \vec{v}\}).$$

Solution: Denote $S_1 = \{3\vec{u}+\vec{v},\ \vec{u}-\vec{v}\}$ and $S_2 = \{\vec{u}, \vec{v}\}$. In order to show $\text{Span}\,(S_1) = \text{Span}\,(S_2)$, it suffices to show that
(1) Every vector in S_1 can be written as a linear combination of the vectors in S_2, and
(2) Every vector in S_2 can be written as a linear combination of the vectors in S_1.

(1) is obvious. (2) is not hard to show, either. We first want to express $\vec{u}$ as a linear combination $u = a(3\vec{u} + \vec{v}) + b(\vec{u} - \vec{v})$ for some a, b. Since $a(3\vec{u} + \vec{v}) + b(\vec{u} - \vec{v}) = (3a + b)\vec{u} + (a - b)\vec{v}$, this means picking a and b so that

$$3a + b = 1 \qquad a - b = 0.$$

One sees easily that the solution is $a = b = \frac{1}{4}$. Thus $\vec{u} = \frac{1}{4}(3\vec{u} + \vec{v}) + \frac{1}{4}(\vec{u} - \vec{v})$. A similar procedure shows that $\vec{v} = \frac{1}{4}(3\vec{u} + \vec{v}) - \frac{3}{4}(\vec{u} - \vec{v})$.

8. Let

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}.$$

(a) Find all eigenvalues of A.

Solution: The eigenvalues of A are the roots of its characteristic polynomial

$$\begin{aligned} \det(A - \lambda I_3) &= \det \left(\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \det \begin{bmatrix} 5-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 1 & 2 & 4-\lambda \end{bmatrix} \\ &= (5-\lambda) \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} = (5-\lambda)\big((1-\lambda)(4-\lambda) - 4\big) \\ &= (5-\lambda)(4 - 5\lambda + \lambda^2 - 4) = -\lambda(\lambda - 5)^2. \end{aligned}$$

Thus A has two eigenvalues: $\lambda = 0$ and $\lambda = 5$.

(b) Find, if possible, an invertible matrix P such that $P^{-1}AP$ is diagonal, or show that no such matrix exists.

Solution: The existence of P is equivalent to saying that 3×3 matrix A is diagonalizable, i.e., has three linearly independent eigenvectors. The theory of eigenspaces provides the tools needed to decide whether or not A is diagonalizable:

- The dimension of each eigenspace is bounded by the multiplicity of the corresponding eigenvalue in the characteristic polynomial. Thus the eigenspace of $\lambda = 0$ has dimension ≤ 1 and the eigenspace of $\lambda = 5$ has dimension ≤ 2.
- A is diagonalizable $\iff$ the dimensions of the eigenspaces add up to 3 $\iff$ the dimension of the each eigenspace *equals* the multiplicity of the eigenvalue in the characteristic polynomial.

We begin with $\lambda = 5$. The eigenspace is the solution space of $(A - 5I_3)\mathbf{x} = \mathbf{0}$. Being homogeneous, we need only row reduce the coefficient matrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix},$$

which has rank 2. By the rank-nullity theorem, the dimension of the nullspace (= the eigenspace of 5) is $3 - 2 = 1$. This is strictly smaller than 2, which is the multiplicity of 5 as a root of the characteristic polynomial. Hence A is not diagonalizable, and no such matrix P exists.

Alternate Solution: The existence of P is equivalent to saying that 3×3 matrix A is diagonalizable, i.e., has three linearly independent eigenvectors. We compute the dimensions of the eigenspaces.

$\lambda = 5$. The eigenspace is the solution space of $(A - 5I_3)\mathbf{x} = \mathbf{0}$. Being homogeneous, we need only row reduce the coefficient matrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix},$$

which has rank 2. By the rank-nullity theorem, the dimension of the nullspace (= the eigenspace of 5) is $3 - 2 = 1$. Hence there is only one linearly independent eigenvector when $\lambda = 5$.

$\lambda = 0$. The eigenspace is the solution space of $(A - 0I_3)\mathbf{x} = A\mathbf{x} = \mathbf{0}$. Being homogeneous, we need only row reduce the coefficient matrix:

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

which has rank 2. By the rank-nullity theorem, the dimension of the nullspace (= the eigenspace of 0) is $3 - 2 = 1$. Hence there is only one linearly independent eigenvector when $\lambda = 0$.

It follows that there are only two linearly independent eigenvectors. So A is not diagonalizable, and no such matrix P exists.

9. Let V be the vector space of polynomials of degree at most 2, and let

$$\mathcal{B} = \{1, x+1, x^2+x+1\},$$

which is a basis for V. Suppose that $T : V \to V$ is a linear transformation, and that the matrix of T relative to $\mathcal{B}$ is

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 2 & 5 & 1 \end{bmatrix}.$$

Find $T(3x^2 + x + 2)$.

Solution: We first compute the coordinate vector of $3x^2 + x + 2$ relative to $\mathcal{B}$. Since

$$\begin{aligned} 3x^2 + x + 2 = 3(x^2+x+1) - 2x - 1 &= 3(x^2+x+1) - 2(x+1) + 1 \\ &= 1(1) - 2(x+1) + 3(x^2+x+1), \end{aligned}$$

the coordinate vector of $3x^2 + x + 2$ is $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$. The coordinate vector of $T(3x^2 + x + 2)$ relative to $\mathcal{B}$ is therefore

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -5 \end{bmatrix}.$$

So $T(3x^2 + x + 2) = 6(1) + 0(x+1) + (-5)(x^2+x+1) = -5x^2 - 5x + 1.$