Solutions to the Algebra problems on the Comprehensive Examination of January 29, 2016

1. [25 points] Let G be a group, let $H \subseteq G$ be a subgroup, and define the set K to be

$$K = \{x \in G \mid Hx = xH\}.$$

   (a) [17 points] Prove that K is a subgroup of G.

   Solution: We must show that K is (1) nonempty, (2) closed under multiplication, and (3) closed under inverses.

   (1) Let e be the identity element of G. Then $He = H = eH$, so $e \in K$.

   (2) Given $a, b \in K$, we have $Ha = aH$ and $Hb = bH$. Thus,

$$H(ab) = (Ha)b = (aH)b = a(Hb) = a(bH) = (ab)H,$$

   and so $ab \in K$.

   (3) Given $a \in K$, we have $Ha = aH$. Thus,

$$\begin{aligned} a^{-1}H &= a^{-1}(He) = a^{-1}\big(H(aa^{-1})\big) = a^{-1}(Ha)a^{-1} = a^{-1}(aH)a^{-1} \\ &= \big((a^{-1}a)H\big)a^{-1} = (eH)a^{-1} = Ha^{-1}. \end{aligned}$$

   Hence, $a^{-1} \in K$. QED

   (b) [8 points] Prove that $H \subseteq K$.

   Solution: Given $a \in H$, we have $Ha = He = H$ and $aH = eH = H$ by the coset criterion. Hence $Ha = aH$, so $a \in K$. QED

2. [25 points] Let G be a group, and let $N \subseteq G$ be a normal subgroup. Prove that

the quotient group G/N is abelian

if and only if

for every $x, y \in G$, we have $xyx^{-1}y^{-1} \in N$.

Solution: ($\Longrightarrow$) Given $x, y \in G$, we have $(Nx)(Ny) = (Ny)(Nx)$ since G/N is abelian. So $Nxy = Nyx$, and hence

$$xyx^{-1}y^{-1} = (xy)(xy)^{-1} \in N$$

by the coset criterion. QED ($\Longrightarrow$)

($\Longleftarrow$) Given $Nx, Ny \in G/N$, we have $x, y \in G$. Therefore, the hypothesis gives

$$(xy)(xy)^{-1} = xyx^{-1}y^{-1} \in N$$

So $Nxy = Nyx$ by the coset criterion, and hence $(Nx)(Ny) = (Ny)(Nx)$. QED

3. [25 points] Consider the group S_{100} of permutations of the set $\{1, 2, 3, \ldots, 100\}$. Let $\sigma \in S_{100}$ be the permutation

$$\sigma = (1\ 3\ 2)(3\ 6)(1\ 4\ 3\ 5)(2\ 3\ 6\ 5\ 4).$$

(a) [8 points] Write σ as a product of **disjoint** cycles.

Solution: $\sigma = (1\ 4)(2\ 5\ 6\ 3)$.

(b) [7 points] Compute the **order** of σ.

Solution: Since σ is a product of disjoint cycles of length 2 and 4, we have $o(\sigma) = \operatorname{lcm}(2, 4) = 4$.

(c) [10 points] For each integer $n = 7, 8, \ldots, 100$, let τ_n be the 4-cycle $\tau_n = (1\ n\ 2\ 4)$. For each such n, decide whether the product $\sigma\tau_n$ is an **even** or **odd** permutation.

Solution: For each $n = 7, \ldots, 100$, τ_n is a 4-cycle and hence is an odd permutation (since 4 is even). Meanwhile, σ is a product of a 2-cycle (odd) and a 4-cycle (also odd). So $\sigma\tau_n$ is

$$\text{odd} + \text{odd} + \text{odd} = \text{odd}$$

4. [25 points] Let R be a ring.

(a) [8 points] Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.

Note: If you use other terms like "closed," "subring," "subgroup," etc., you must fully define those terms as well.

Solution: $I \subseteq R$ is said to be an ideal of R if

i. I is nonempty.

ii. for every $x, y \in I$, we have $x - y \in I$.

iii. for every $a \in R$ and $b \in I$, we have $ab, ba \in I$.

(b) [17 points] Let S be another ring, and let $\phi : R \to S$ be a ring homomorphism. Let $I \subseteq R$ be an ideal of R, and define

$$J = \{x \in I \mid \phi(x) = 0_S\},$$

where 0_S denotes the zero element of S. Give a complete proof that J is an ideal of R.

Solution: [*Caution!* J is *not* the kernel of ϕ, so we cannot just use the theorem that states $\ker(\phi)$ is an ideal.] We first note that $J \subseteq I \subseteq R$.

i. We have $0_R \in I$ and $\phi(0_R) = 0_S$, so $0_R \in J$, and hence $J \neq \varnothing$.

ii. Given $x, y \in J$, then $x, y \in I$, so $x - y \in I$. Moreover,

$$\phi(x - y) = \phi(x) - \phi(y) = 0_S - 0_S = 0_S.$$

So $x - y \in J$.

iii. Given $r \in R$ and $j \in J \subseteq I$. Then $rj, jr \in I$. Furthermore,

$$\phi(rj) = \phi(r)\phi(j) = \phi(r) \cdot 0_S = 0_S, \quad \text{and}$$
$$\phi(jr) = \phi(j)\phi(r) = 0_S \cdot \phi(r) = 0_S,$$

so $rj, jr \in J$. QED