Solutions to the Analysis problems on the Comprehensive Examination of January 29, 2016

1. [25 points]

   (a) [5 points] What does it mean to say that a sequence (a_n) of real numbers *converges* to the real number a?

   Solution: (a_n) converges to a if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - a| < \epsilon$.

   Alternative Solution: (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in V_\epsilon(a)$. (Note, however, that this definition is not useful in part (c).)

   (b) [5 points] What does it mean to say that a sequence (a_n) of real numbers is a *Cauchy sequence*?

   Solution: (a_n) is a Cauchy sequence if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, n \geq N$, then $|a_n - a_m| < \epsilon$.

   (c) [15 points] Let (a_n) be a convergent sequence of real numbers. Prove that (a_n) is a Cauchy sequence.

   Solution: This proof utilizes the triangle inequality. Suppose that (a_n) converges to a. Pick an arbitrary $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that if $k \geq N$, then $|a_k - a| < \epsilon/2$. Suppose $m, n \geq N$. Then

   $$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

   Because ϵ is arbitrary, (a_n) is Cauchy.

2. [25 points]

   (a) [5 points] State the ϵ/δ definition of what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *continuous* at c.

   Solution: $f : \mathbb{R} \to \mathbb{R}$ is continuous at c if, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

   (b) [20 points] Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions such that g is continuous at c and f is continuous at $g(c)$. Prove that the composite function $f \circ g$, defined by

   $$(f \circ g)(x) = f(g(x)),$$

   is continuous at c. (If your proof does not use the definition given in part (a), be sure to state clearly the property of continuity you are using and explain how it is being used.)

   Solution: We will use the definition in part (a). Pick an arbitrary $\epsilon > 0$. Since f is continuous at $g(c)$, by definition there is $\delta' > 0$ such that $|y - g(c)| < \delta' \Rightarrow$

$|f(y) - f(g(c))| < \epsilon$. Similarly, since g is continuous at c, there is $\delta > 0$ such that $|x - c| < \delta \Rightarrow |g(x) - g(c)| < \delta'$. Combining these two inequalities, we have

$$|x - c| < \delta \ \Rightarrow \ |g(x) - g(c)| < \delta' \ \Rightarrow \ |f(g(x)) - f(g(c))| < \epsilon,$$

which is exactly what we need to show $f \circ g$ is continuous at c.

3. [30 points]

(a) [5 points] What does it mean to say that a power series $\sum_{n=0}^{\infty} c_n x^n$ *converges uniformly* on an interval $[a, b]$?

Solution: $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly on $[a, b]$ if the sequence $s_n(x)$ of partial sums defined by

$$s_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

converges uniformly on $[a, b]$.

Here, when we say that $\{s_n(x)\}$ converges uniformly on $[a, b]$ we mean that there is a function $s : [a, b] \to \mathbb{R}$ that has the property that for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $x \in [a, b]$ and $n \geq N$ it follows that $|s_n(x) - s(x)| < \epsilon$.

(b) [15 points] Prove that the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges uniformly on the interval $[-r, r]$ for any $0 < r < 1$. (You may use the Weierstrass M-Test if you state the test precisely and explain clearly how it applies.)

Solution: First, let us state the Weierstrass M-Test. Let $f_n : A \to \mathbb{R}$ be a function defined on A for all $n \in \mathbb{N}$ and let $M_n > 0$ be a real number such that $\forall\ x \in A$, $|f_n(x)| \leq M_n$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

Now fix an arbitrary $r \in (0, 1)$. Let $x \in [-r, r]$. Then $\forall n \in \mathbb{N}$,

$$\left| \frac{x^n}{n} \right| = \frac{|x^n|}{|n|} = \frac{|x^n|}{n} = \frac{|x|^n}{n} \leq \frac{r^n}{n} \leq r^n.$$

(Note that this is a fixed number for each n since r is fixed.) Since $0 < r < 1$, the geometric series $\sum_{n=1}^{\infty} r^n$ converges. Therefore, by the Weierstrass M-Test with $M_n = r^n$, $\sum_{n=1}^{\infty} x^n/n$ converges uniformly on $[-r, r]$. Finally, because r is arbitrary, $\sum_{n=1}^{\infty} x^n/n$ converges uniformly on $[-r, r]$ for all $r \in (0, 1)$.

(c) [10 points] Does the power series in part (b) converge uniformly on $[-1, 1]$? Justify your answer.

Solution: No. At $x = 1$, $\sum_{n=1}^{\infty} x^n/n = \sum_{n=1}^{\infty} 1/n$ is a p-series with $p = 1$, and is therefore divergent (recall that it is known as the harmonic series). Therefore the series does not converge uniformly on $[-1, 1]$, because uniform convergence implies pointwise convergence, yet the latter fails at $x = 1$.

4. [20 points] Do EITHER part (a) OR part (b), NOT BOTH.

(a) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ that is not Riemann-integrable on the interval $[0,1]$. Justify your answer based on the definition of Riemann integration.

Solution: Let us first review the definition of Riemann integrability on $[0,1]$. Let f be a bounded function defined on the interval $[0,1]$, and let $\mathcal{P}$ be the collection of all possible partitions of $[0,1]$. Then f is Riemann-integrable if and only if

$$\sup\{L(f,P) \mid P \in \mathcal{P}\} = \inf\{U(f,P) \mid P \in \mathcal{P}\}, \tag{1}$$

where for a partition $P = \{0 = x_0 < \cdots < x_n = 1\}$ of $[0,1]$,

$$L(f,P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}), \quad m_k = \inf\{f(x) \mid x_{k-1} \le x \le x_k\}$$
$$U(f,P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}), \quad M_k = \sup\{f(x) \mid x_{k-1} \le x \le x_k\}.$$

To violate (1), we use Dirichlet's function:

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

Let P be a partition of $[0,1]$. Because both the rationals and the irrationals are dense in $\mathbb{R}$, we have $m_k = 0$ and $M_k = 1$ for all k, which easily implies $L(g,P) = 0$ and $U(g,P) = 1$. Since this is true for all partitions P, we get

$$\sup\{L(g,P) \mid P \in \mathcal{P}\} = 0 < 1 = \inf\{U(f,P) \mid P \in \mathcal{P}\},$$

so g is not Riemann integrable on $[0,1]$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f has a local maximum at c. Prove that $f'(c) = 0$.

Solution: Because $f(c)$ is a local maximum, we may choose $a, b \in \mathbb{R}$ such that $a < c < b$ and $f(x) \le f(c)$ for all $x \in (a,b)$. By definition,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

For $x \in (a,c)$, we have $f(x) - f(c) \le 0$ and $x - c < 0$, so $\frac{f(x)-f(c)}{x-c} \ge 0$, which means that $f'(c) = \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \ge 0$. Similarly, for $x \in (c,b)$, $f(x) - f(c) \le 0$ but $x - c > 0$, so $\frac{f(x)-f(c)}{x-c} \le 0$, which means that $f'(c) = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \le 0$. Combining these two inequalities, we have $0 \le f'(c) \le 0$, so $f'(c) = 0$, as desired.

Alternate Solution: See Theorem 5.2.6 in Abbott's *Understanding Analysis* 2e.