**Amherst College**
**Department of Mathematics and Statistics**

# Comprehensive and Honors Qualifying Examination

## ◁ Algebra ▷

## January 2017

Number: _____

Senior: _____

Junior: _____

**Read This First:**

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space, and indicate whether you are a junior or a senior.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- The Algebra Exam consists of Questions 1–4 that total to 100 points.

**For Department Use Only:**

Grader #1: _________________

Grader #2: _________________

1. [25 points] Let $G$ be a group, let $H \subseteq G$ be a subgroup, and let $N \subseteq G$ be a **normal** subgroup. Define
$$NH = \{nh : n \in N, h \in H\}.$$
Prove that $NH$ is a subgroup of $G$.

2. [17 points] Let $G$ be a finite group, and suppose that there is an element $a \in G$ with the property that $a^2 = a^{-1}$ but $a$ is not the identity. Prove that the order of $G$ is divisible by 3.

3. Consider the group $S_8$ of permutations of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.
   (a) [10 points] Find an element of $S_8$ of order 15.
   (Don't forget to explain or prove why it has order 15.)
   (b) [15 points] Prove that there are no **odd** permutations in $S_8$ of order 15.

4. Let $R$ be a ring.
   (a) [8 points] Define what it means for a subset $I \subseteq R$ to be an **ideal** of $R$.
   If you use any other technical terms like "closed," "subring," "group," "subgroup," etc., you must fully define those terms as well.
   (b) [25 points] Suppose that $R$ is commutative and has a multiplicative identity 1. Let $I \subseteq J \subseteq R$ be ideals, and suppose that the quotient ring $R/I$ is a field.

   If $I \subsetneq J$, prove that $1 \in J$.

   (In fact, it is a Theorem from Math 350 that $J = R$ in this case, but you are only being asked to prove that $1 \in J$. In particular, however, you may **not** quote the $J = R$ theorem.)