Amherst College
Department of Mathematics and Statistics

Comprehensive and Honors Qualifying Examination

◁ Analysis Solutions ▷

January 2017

Number: _____

Senior: _____

Junior: _____

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space, and indicate whether you are a junior or a senior.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- The Analysis Solutions Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: _________________

Grader #2: _________________

1. Suppose we have nonempty subsets $A, B \subseteq \mathbb{R}$ that are bounded above. Define

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

   (a) [10 points] Prove that $A + B$ is bounded above.

   Solution: Since A and B are both nonempty and bounded above, we know from the Axiom of Completeness that their supremums exist. Let $\mathrm{s} = \sup(A)$ and $t = \sup(B)$. We aim to show that $s + t$ is an upper bound for $A + B$.

   Let $x \in A + B$. There exists some $a \in A$ and $b \in B$ such that $x = a + b$. Since s and t are upper bounds for A and B respectively, we know that $a \leq s$ and $b \leq t$. Thus $x = a + b \leq s + t$ and $A + B$ is bounded above by $s + t$. □

   (b) [15 points] Prove that $\sup(A + B) \leq \sup(A) + \sup(B)$.

   Solution: From part (a), we know that $A+B$ is bounded and clearly nonempty so by the Axiom of Completeness, $\sup(A + B)$ exists. Further, from part (a) we know that $\sup(A)+\sup(B)$ is an upper bound for $A+B$. Since $\sup(A+B)$ is the smallest upper bound for $A + B$, it must be that $\sup(A + B) \leq \sup(A) + \sup(B)$. □

2. (a) [10 points] State the ϵ-N definition of what it means for a sequence (a_n) of real numbers to converge to $a \in \mathbb{R}$.

   Solution: A sequence of real numbers is said to converge to a real number a if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a - a_n| < \epsilon$.

   Note: One could also give the Topological Version of Convergence. A sequence of real numbers is said to converge to a real number a if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $a_n \in V_\epsilon(a)$.

   (b) [15 points] Assume that sequences (a_n) and (b_n) converge to real numbers a and b respectively. In other words, $(a_n) \to a$ and $(b_n) \to b$. Use the definition given in part (a) to prove that $(a_n + b_n) \to a + b$.

   Solution: Let $\epsilon > 0$ be given. By definition of convergence, since $(a_n) \to a$ we know that there exists an $N_1 \in \mathbb{N}$ such that whenever $n \geq N_1$ it follows that $|a - a_n| < \frac{\epsilon}{2}$. Similarly, since $(b_n) \to b$ we know that there exists an $N_2 \in \mathbb{N}$ such that whenever $n \geq N_2$ it follows that $|b - b_n| < \frac{\epsilon}{2}$. Letting $N = \max\{N_1, N_2\}$ gives us that, whenever $n \geq N$ it follows that

$$\begin{aligned}
|(a + b) - (a_n + b_n)| &= |(a - a_n) + (b - b_n)| \\
&\leq |a - a_n| + |b - b_n| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

   where the last inequality follows form the fact that $n \geq N_1$ and $n \geq N_2$. Thus, by the definition of convergence $(a_n + b_n) \to a + b$. □

(c) [10 points] In the limit notation used in calculus, the result of Problem 2(b) can be stated as the implication

(1) $$\text{if } \lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b, \text{ then } \lim_{n\to\infty} (a_n + b_n) = a + b.$$

Suppose that we have k sequences $(a_{1,n}), (a_{2,n}), \ldots (a_{k,n})$. Use (1) and induction on $k \geq 1$ to prove that if $\lim_{n\to\infty} a_{1,n} = a_1, \ldots, \lim_{n\to\infty} a_{k,n} = a_k$, then

$$\lim_{n\to\infty} (a_{1,n} + \cdots + a_{k,n}) = a_1 + \cdots + a_k.$$

Solution: The base case, when $k = 1$ is trivially true.

Next, assume that the statement is true for $k = \ell$ sequences and that we have $\ell + 1$ sequences $(a_{1,n}), (a_{2,n}), \ldots (a_{\ell+1,n})$ such that $\lim_{n\to\infty} a_{i,n} = a_i$ for each $1 \leq i \leq \ell + 1$. In this case we have that

$$\begin{aligned}
\lim_{n\to\infty} (a_{1,n} + \cdots + a_{\ell+1,n}) &= \lim_{n\to\infty} \Big[(a_{1,n} + \cdots a_{\ell,n}) + a_{\ell+1,n}\Big] \\
&= \lim_{n\to\infty} (a_{1,n} + \cdots a_{\ell,n}) + \lim_{n\to\infty} a_{\ell+1,n} \\
&= (a_1 + \cdots + a_\ell) + a_{\ell+1} = a_1 + \cdots + a_\ell + a_{\ell+1},
\end{aligned}$$

where the second equality follows from 2(b) and the third equality follows from the inductive assumption. □

3. (a) [5 points] Suppose that $A \subseteq \mathbb{R}$ and for every $n \in \mathbb{N}$ we have a function $f_n : A \to \mathbb{R}$. Define what it means for (f_n) to converge *pointwise* on A to a function $f : A \to \mathbb{R}$.

Solution: The sequence of functions (f_n) converges pointwise on A to f if for every $\epsilon > 0$ and $x \in A$ there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|f_n(x) - f(x)| < \epsilon$.

(b) [10 points] Using the notation of part (a), define what it means for (f_n) to converge *uniformly* on A to f.

Solution: The sequence of functions (f_n) converges uniformly on A to f if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ and $x \in A$ it follows that $|f_n(x) - f(x)| < \epsilon$.

Note: The difference between pointwise and uniform convergence is that in uniform convergence the choice of $N \in \mathbb{N}$ can be made independently of the value of $x \in A$.

(c) [15 points] Suppose that $g : A \to \mathbb{R}$ is a bounded function. For each $n \in \mathbb{N}$, define

$$f_n(x) = \frac{g(x)}{n}.$$

There is a function f such that (f_n) converges uniformly to f. Find f and prove that the convergence is uniform.

Solution: Let $\epsilon > 0$ be given. Since $g(x)$ is bounded on A, there exists an $M > 0$ such that for every $x \in A$ we have that $|g(x)| \leq M$. By the archimedean property, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{M}$. Let $f(x)$ be the function that is uniformly zero on A. For every $n \geq N$ and $x \in A$ it follows that

$$|f_n(x) - f(x)| = |f_n(x) - 0| = \left|\frac{g(x)}{n}\right| = \frac{|g(x)|}{n} \leq \frac{M}{n} \leq \frac{M}{N} < \frac{M}{\left(\frac{\epsilon}{M}\right)} = \epsilon.$$

Therefore, f_n converges uniformly to f on A. □

4. [10 points] Suppose that we have open sets $O_\lambda \subseteq \mathbb{R}$ for all λ in some index set Λ. Prove that the union $O = \displaystyle\bigcup_{\lambda \in \Lambda} O_\lambda$ is an open set.

Solution: Let $a \in O = \displaystyle\bigcup_{\lambda \in \Lambda} O_\lambda$. By definition there exists a $\lambda_0 \in \Lambda$ such that $a \in O_{\lambda_0}$. Since O_{λ_0} is open, there exists an $\epsilon > 0$ such that the epsilon neighborhood of a, $V_\epsilon(a)$, is contained in O_{λ_0}. Therefore, $V_\epsilon(a) \subseteq O_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda = O$ and O is open by definition. □