Comprehensive and Honors Qualifying Examination
\< Analysis >
January 2017

Number: _____
Senior: _____
Junior: _____

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.

- Write your number (not your name) in the above space, and indicate whether you are a junior or a senior.

- For any given problem, you may use the back of the previous page for scratch work. Put your final answers in the spaces provided.

- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.

- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

- The Analysis Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: ______________
Grader #2: ______________
1. Suppose we have nonempty subsets $A, B \subseteq \mathbb{R}$ that are bounded above. Define

$$A + B = \{a + b \mid a \in A, b \in B\}.$$ 

(a) [10 points] Prove that $A + B$ is bounded above.

(b) [15 points] Prove that $\sup(A + B) \leq \sup(A) + \sup(B)$.

2. (a) [10 points] State the $\epsilon$-$N$ definition of what it means for a sequence $(a_n)$ of real numbers to converge to $a \in \mathbb{R}$.

(b) [15 points] Assume that sequences $(a_n)$ and $(b_n)$ converge to real numbers $a$ and $b$ respectively. In other words, $(a_n) \to a$ and $(b_n) \to b$. Use the definition given in part (a) to prove that $(a_n + b_n) \to a + b$.

(c) [10 points] In the limit notation used in calculus, the result of Problem 2(b) can be stated as the implication

$$\text{if } \lim_{n \to \infty} a_n = a \text{ and } \lim_{n \to \infty} b_n = b, \text{ then } \lim_{n \to \infty} (a_n + b_n) = a + b. \quad (1)$$

Suppose that we have $k$ sequences $(a_{1,n})$, $(a_{2,n})$, \ldots, $(a_{k,n})$. Use (1) and induction on $k \geq 1$ to prove that if $\lim_{n \to \infty} a_{1,n} = a_1$, \ldots, $\lim_{n \to \infty} a_{k,n} = a_k$, then

$$\lim_{n \to \infty} (a_{1,n} + \cdots + a_{k,n}) = a_1 + \cdots + a_k.$$

3. (a) [5 points] Suppose that $A \subseteq \mathbb{R}$ and for every $n \in \mathbb{N}$ we have a function $f_n : A \to \mathbb{R}$. Define what it means for $(f_n)$ to converge pointwise on $A$ to a function $f : A \to \mathbb{R}$.

(b) [10 points] Using the notation of part (a), define what it means for $(f_n)$ to converge uniformly on $A$ to $f$.

(c) [15 points] Suppose that $g : A \to \mathbb{R}$ is a bounded function. For each $n \in \mathbb{N}$, define

$$f_n(x) = \frac{g(x)}{n}.$$ 

There is a function $f$ such that $(f_n)$ converges uniformly to $f$. Find $f$ and prove that the convergence is uniform.

4. [10 points] Suppose that we have open sets $O_{\lambda} \subseteq \mathbb{R}$ for all $\lambda$ in some index set $\Lambda$. Prove that the union $O = \bigcup_{\lambda \in \Lambda} O_{\lambda}$ is an open set.