Amherst College
Department of Mathematics and Statistics

Comprehensive and Honors Qualifying Examination

◁ Analysis ▷

January 2017

Number: _____

Senior: _____

Junior: _____

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space, and indicate whether you are a junior or a senior.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- The Analysis Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: _________________

Grader #2: _________________

1. Suppose we have nonempty subsets $A, B \subseteq \mathbb{R}$ that are bounded above. Define

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

   (a) [10 points] Prove that $A + B$ is bounded above.

   (b) [15 points] Prove that $\sup(A + B) \leq \sup(A) + \sup(B)$.

2. (a) [10 points] State the ϵ-N definition of what it means for a sequence (a_n) of real numbers to converge to $a \in \mathbb{R}$.

   (b) [15 points] Assume that sequences (a_n) and (b_n) converge to real numbers a and b respectively. In other words, $(a_n) \to a$ and $(b_n) \to b$. Use the definition given in part (a) to prove that $(a_n + b_n) \to a + b$.

   (c) [10 points] In the limit notation used in calculus, the result of Problem 2(b) can be stated as the implication

$$\text{(1)} \qquad \text{if } \lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b, \text{ then } \lim_{n\to\infty}(a_n + b_n) = a + b.$$

   Suppose that we have k sequences $(a_{1,n}), (a_{2,n}), \ldots (a_{k,n})$. Use (1) and induction on $k \geq 1$ to prove that if $\lim_{n\to\infty} a_{1,n} = a_1, \ldots, \lim_{n\to\infty} a_{k,n} = a_k$, then

$$\lim_{n\to\infty}(a_{1,n} + \cdots + a_{k,n}) = a_1 + \cdots + a_k.$$

3. (a) [5 points] Suppose that $A \subseteq \mathbb{R}$ and for every $n \in \mathbb{N}$ we have a function $f_n : A \to \mathbb{R}$. Define what it means for (f_n) to converge *pointwise* on A to a function $f : A \to \mathbb{R}$.

   (b) [10 points] Using the notation of part (a), define what it means for (f_n) to converge *uniformly* on A to f.

   (c) [15 points] Suppose that $g : A \to \mathbb{R}$ is a bounded function. For each $n \in \mathbb{N}$, define

$$f_n(x) = \frac{g(x)}{n}.$$

   There is a function f such that (f_n) converges uniformly to f. Find f and prove that the convergence is uniform.

4. [10 points] Suppose that we have open sets $O_\lambda \subseteq \mathbb{R}$ for all λ in some index set Λ. Prove that the union $O = \bigcup_{\lambda\in\Lambda} O_\lambda$ is an open set.