**Solutions to the Multivariable Calculus and Linear Algebra problems on the Comprehensive Examination of January 27, 2017**

1. (a) [15 points] Find an equation of form $ax+by+cz=d$ for the plane passing through $(-2,-1,4)$ that is perpendicular to the line with parametric equations $x=2t,\ y=3t-1,\ z=5-t$.

   **Solution:** The plane should have the same normal vector as the line parameterized as $(x,y,z)=(2t,3t-1,5-t)=t(2,3,-1)+(0,-1,5)$. Therefore, the plane has normal vector $(2,3,-1)$. It must additionally pass through $(-2,-1,4)$, hence, its equation is given by

$$\begin{aligned}&2(x-(-2))+3(y-(-1))+(-1)(z-4)=0\\ \Leftrightarrow\ &2x+3y-z=-11.\end{aligned}$$

   (b) [15 points] Find an equation for the tangent plane to the sphere $x^2+y^2+z^2=3$ at the point $(1,-1,1)$.

   **Solution:** The sphere $x^2+y^2+z^2=3$ is a level surface of the function $F(x,y,z)=x^2+y^2+z^2$. We have

$$F_x(x,y,z)=2x, F_y(x,y,z)=2y, F_z(x,y,z)=2z,$$

   and hence

$$F_x(1,-1,1)=2, F_y(1,-1,1)=-2, F_z(1,-1,1)=2.$$

   The tangent plane at $(1,-1,1)$ is thus given by

$$\begin{aligned}&F_x(1,-1,1)(x-1)+F_y(1,-1,1)(y-(-1))+F_z(1,-1,1)(z-1)=0\\ \Leftrightarrow\ &2(x-1)+-2(y+1)+2(z-1)=0\\ \Leftrightarrow\ &2x-2y+2z=6\\ \Leftrightarrow\ &x-y+z=3.\end{aligned}$$

2. Let $f(x,y)=\begin{cases}\dfrac{3x^3-5y^3}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$

   (a) [15 points] Compute $f_x(0,0)$ and $f_y(0,0)$.

   **Solution:** First observe that when $h\neq 0$, we have

$$f(h,0)=\frac{3h^3-0}{h^2+0}=3h,\quad f(0,h)=\frac{0-5h^3}{0+h^2}=-5h.$$

   Then we use the definition of partial derivative to obtain

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{3h-0}{h}=\lim_{h\to 0}3=3$$

   and

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{-5h-0}{h}=\lim_{h\to 0}-5=-5.$$

(b) [10 points] Is $f$ is continuous at $(0,0)$? Justify your answer.

**Solution:** Recall that $f$ is continuous at $(0,0)$ iff $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$. Here, $f(0,0) = 0$, so it suffices to show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. In polar coordinates, $x = r\cos\theta, y = r\sin\theta$, and $(x,y)\to(0,0)$ becomes $r\to 0$. Then

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{3x^3-5y^3}{x^2+y^2} &= \lim_{r\to 0} \frac{3r^3\cos^3\theta - 5r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \\ &= \lim_{r\to 0} \frac{r^3(3\cos^3\theta - 5\sin^3\theta)}{r^2(\sin^2\theta+\cos^2\theta)} \\ &= \lim_{r\to 0} r(3\cos^3\theta - 5\sin^3\theta) = 0\end{aligned}$$

as desired, where we have used $\sin^2\theta + \cos^2\theta = 1$, and that $3\cos^3\theta - 5\sin^3\theta$ is bounded.

3. [20 points] Find the volume of the solid region bounded by the surfaces $y = 0$, $y = 3$, $z = x^2$, and $z = 2 - x^2$.

**Solution:** The curves $z = x^2$ and $z = 2-x^2$ intersect when $x^2 = 2 - x^2 \Leftrightarrow 2x^2 = 2 \Leftrightarrow x = \pm 1$. For $-1 \le x \le 1$, the curve $z = 2 - x^2$ sits above the curve $z = x^2$. Therefore, the volume is given by

$$\begin{aligned}V &= \int_{y=0}^{3}\int_{x=-1}^{1} \left((2-x^2)-x^2\right)\,dx\,dy = \int_{y=0}^{3}\int_{x=-1}^{1}\left(2-2x^2\right)\,dx\,dy \\ &= \int_{y=0}^{3}\left(2\int_{x=0}^{1}\left(2-2x^2\right)\,dx\right)\,dy = 2\int_{y=0}^{3}\left[2x - \frac{2}{3}x^3\right]_{x=0}^{x=1}\,dy \\ &= 2\left(2-\frac{2}{3}\right)\int_{y=0}^{3}\,dy = 2\left(2-\frac{2}{3}\right)3 = 8.\end{aligned}$$

4. [25 points] Show that the line integral

$$\int_C z^2\,dx \;+\; 2y\,dy \;+\; 2xz\,dz$$

depends only on the endpoints of the path $C$ and not on the path taken between those endpoints.

**Solution:** Let $\boldsymbol{x} = (x,y,z)$, and let $F(\boldsymbol{x}) = (z^2, 2y, 2xz)$. If we let $f(\boldsymbol{x}) = z^2x + y^2$, then the gradient of $f$ satisfies $\nabla f(\boldsymbol{x}) = (f_x, f_y, f_z) = (z^2, 2y, 2xz) = F(\boldsymbol{x})$. That is, $F$ is a conservative vector field with potential function $f$. If $C$ is given by $\boldsymbol{r}(t)$, where $a \le t \le b$, then

$$\int_C z^2\,dx \;+\; 2y\,dy \;+\; 2xz\,dz = \int_C F\cdot d\boldsymbol{r} = \int_C \nabla f\cdot d\boldsymbol{r} = f(\boldsymbol{r}(b)) - f(\boldsymbol{r}(a))$$

by the Fundamental Theorem for Line Integrals, and is therefore independent of path, and depends only on the end points $\boldsymbol{r}(a)$ and $\boldsymbol{r}(b)$.

5. Let $V$ denote the vector space of polynomials of degree less than or equal to 2 with real coefficients. Let $T : V \to \mathbb{R}$ be a linear transformation.

   (a) [5 points] Explain what is meant by the *kernel*, or *null space*, of $T$.
   **Solution:** The set
   $$N(T) = \{\mathbf{x} \in V \mid T(\mathbf{x}) = 0\}.$$

   (b) [15 points] Prove that the kernel of $T$ is a subspace of $V$.
   **Solution:** Given $\mathbf{x}_1, \mathbf{x}_2 \in N(T)$ arbitrary:
   $$T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2) = 0 + 0 = 0.$$
   So $\mathbf{x}_1 + \mathbf{x}_2 \in N(T)$ and so $N(T)$ is closed under addition.
   Given $\mathbf{x} \in N(T)$ and $c \in \mathbb{R}$ arbitrary:
   $$T(c\mathbf{x}) = cT(\mathbf{x}) = c0 = 0.$$
   So $c\mathbf{x} \in N(T)$ and so $N(T)$ is closed under scalar multiplication.
   Since $T(0) = 0$ we have $0 \in N(T)$.
   Therefore $N(T)$ is a subspace of $V$.

   (c) [10 points] What are the possible values of the nullity (that is, the dimension of the kernel) of $T$? Justify your answer.
   **Solution:** By the Rank-Nullity Theorem, the nullity of $T$ is equal to $\dim(V) - \text{rank}(T)$. Since the target vector space is one-dimensional, the rank is either 0 or 1. The dimension of $V$ is 3. Therefore, the nullity of $T$ is equal to either 2 or 3.

6. [20 points] Suppose that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for a vector space $U$. Is the set
   $$\{\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 + 2\mathbf{u}_2\}$$
   also a basis for $U$? Justify your answer.

   **Solution:** To decide if the set is linearly independent, suppose that
   $$a(\mathbf{u}_1 - \mathbf{u}_2) + b(\mathbf{u}_1 + 2\mathbf{u}_2) = \mathbf{0}.$$
   Rearranging we have
   $$(a + b)\mathbf{u}_1 + (-a + 2b)\mathbf{u}_2 = \mathbf{0}.$$
   Since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis, it is linearly independent and so
   $$a + b = 0, \quad -a + 2b = 0.$$
   Adding these two equations, we get $3b = 0$ and so $b = 0$. Therefore, also $a = 0$. So the set
   $$\{\mathbf{u}_1 - \mathbf{u}_2\, \mathbf{u}_1 + 2\mathbf{u}_2\}$$
   is linearly independent.

   We know that $\dim(U) = 2$ since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis. Therefore, the linearly independent set $\{\mathbf{u}_1 - \mathbf{u}_2\, \mathbf{u}_1 + 2\mathbf{u}_2\}$ must also be a basis.

7. (a) [5 points] Explain what it means to say that a real number $\lambda$ is an *eigenvalue* of an $n \times n$ matrix $A$.

**Solution:** It means that there is some $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \neq \mathbf{0}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

(b) [10 points] Calculate the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & -1 \end{bmatrix}.$$

**Solution:** The characteristic polynomial is

$$\det \begin{bmatrix} 2-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{bmatrix}$$

which gives

$$(2-\lambda)((1-\lambda)(-1-\lambda)-3) = (2-\lambda)((\lambda^2-4) = -(\lambda-2)^2(\lambda+2).$$

Therefore, the eigenvalues of $A$ are $\lambda = 2$ (with multiplicity 2) and $\lambda = -2$ (with multiplicity 1).

(c) [15 points] Decide if $A$ is diagonalizable or not. Justify your answer.

**Solution:** To find eigenvectors for $\lambda = 2$, we solve

$$(A - 2I)\mathbf{x} = \mathbf{0}$$

which gives

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 3 & -3 \end{bmatrix} \mathbf{x} = \mathbf{0}.$$

In reduced row echelon form, this linear system is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

which has solutions

$$\mathbf{x} = \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

This eigenspace is 1-dimensional. Since the multiplicity of the eigenvalue 2 is 2, this means that $A$ is **not** diagonalizable.

8. (a) [5 points] Let $U$ and $W$ be vector spaces. Explain what it means to say that a linear transformation $S : U \to W$ is *invertible*.

   **Solution:** It means that there is some linear transformation $S^{-1} : W \to U$ such that $SS^{-1} = I_W$, the identity on $W$, and $S^{-1}S = I_U$, the identity on $U$. (Alternatively, one could also explain invertibility in terms of injectivity and surjectivity of $S$.)

   (b) [5 points] Let $T : V \to V$ be a linear transformation, and let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for the vector space $V$. Suppose that the matrix of $T$ with respect to $\alpha$ is

$$\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

   Explain how you know that $T$ is invertible.

   **Solution:** The matrix above has determinant $5 \neq 0$ so is an invertible matrix. Therefore $T$ is invertible.

   (c) [10 points] Calculate $T^{-1}(\mathbf{v}_1)$. Write your answer as a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

   **Solution:** The matrix for $T^{-1}$ is given by

$$\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}.$$

   Therefore

$$T^{-1}(\mathbf{v}_1) = \frac{1}{5}\mathbf{v}_1 - \frac{2}{5}\mathbf{v}_2.$$