1. (a) [15 points] Find an equation of form $ax + by + cz = d$ for the plane passing through $(-2, -1, 4)$ that is perpendicular to the line with parametric equations $x = 2t$, $y = 3t - 1$, $z = 5 - t$.

**Solution:** The plane should have the same normal vector as the line parameterized as $(x, y, z) = (2t, 3t - 1, 5 - t) = t(2, 3, 1) + (0, -1, 5)$. Therefore, the plane has normal vector $(2, 3, 1)$. It must additionally pass through $(-2, -1, 4)$, hence, its equation is given by

$$2(x - (-2)) + 3(y - (-1)) + (z - 4) = 0$$

$$2x + 3y - z = -11.$$  

(b) [15 points] Find an equation for the tangent plane to the sphere $x^2 + y^2 + z^2 = 3$ at the point $(1, -1, 1)$.

**Solution:** The sphere $x^2 + y^2 + z^2 = 3$ is a level surface of the function $F(x, y, z) = x^2 + y^2 + z^2$. We have

$$F_x(x, y, z) = 2x, F_y(x, y, z) = 2y, F_z(x, y, z) = 2z,$$

and hence

$$F_x(1, -1, 1) = 2, F_y(1, -1, 1) = -2, F_z(1, -1, 1) = 2.$$  

The tangent plane at $(1, -1, 1)$ is thus given by

$$F_x(1, -1, 1)(x - 1) + F_y(1, -1, 1)(y - (-1)) + F_z(1, -1, 1)(z - 1) = 0$$

$$2(x - 1) - 2(y + 1) + 2(z - 1) = 0$$

$$2x - 2y + 2z = 6$$

$$x - y + z = 3.$$  

2. Let $f(x, y) = \begin{cases} 3x^3 - 5y^3 & \text{if } (x, y) \neq (0, 0) \\ x^2 + y^2 & \text{if } (x, y) = (0, 0) \end{cases}$

(a) [15 points] Compute $f_x(0, 0)$ and $f_y(0, 0)$.

**Solution:** First observe that when $h \neq 0$, we have

$$f(h, 0) = \frac{3h^3 - 0}{h^2 + 0} = 3h, \quad f(0, h) = \frac{0 - 5h^3}{0 + h^2} = -5h.$$  

Then we use the definition of partial derivative to obtain

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{3h - 0}{h} = \lim_{h \to 0} 3 = 3$$

and

$$f_y(0, 0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{-5h - 0}{h} = \lim_{h \to 0} -5 = -5.$$
4. [25 points] Show that the line integral
\[ \int_C z^2 \, dx + 2y \, dy + 2xz \, dz \]
depends only on the endpoints of the path $C$ and not on the path taken between those endpoints.

Solution: Let $\mathbf{x} = (x, y, z)$, and let $F(\mathbf{x}) = (z^2, 2y, 2xz)$. If we let $f(\mathbf{x}) = z^2x + y^2$, then the gradient of $f$ satisfies $\nabla f(\mathbf{x}) = (f_x, f_y, f_z) = (z^2, 2y, 2xz) = F(\mathbf{x})$. That is, $F$ is a conservative vector field with potential function $f$. If $C$ is given by $\mathbf{r}(t)$, where $a \leq t \leq b$, then
\[ \int_C z^2 \, dx + 2y \, dy + 2xz \, dz = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \]
by the Fundamental Theorem for Line Integrals, and is therefore independent of path, and depends only on the end points $\mathbf{r}(a)$ and $\mathbf{r}(b)$.
5. Let $V$ denote the vector space of polynomials of degree less than or equal to 2 with real coefficients. Let $T : V \to \mathbb{R}$ be a linear transformation.

(a) [5 points] Explain what is meant by the kernel, or null space, of $T$.

Solution: The set 

$$ N(T) = \{ x \in V \mid T(x) = 0 \}. $$

(b) [15 points] Prove that the kernel of $T$ is a subspace of $V$.

Solution: Given $x_1, x_2 \in N(T)$ arbitrary:

$$ T(x_1 + x_2) = T(x_1) + T(x_2) = 0 + 0 = 0. $$

So $x_1 + x_2 \in N(T)$ and so $N(T)$ is closed under addition.

Given $x \in N(T)$ and $c \in \mathbb{R}$ arbitrary:

$$ T(cx) = cT(x) = c0 = 0. $$

So $cx \in N(T)$ and so $N(T)$ is closed under scalar multiplication.

Since $T(0) = 0$ we have $0 \in N(T)$.

Therefore $N(T)$ is a subspace of $V$.

(c) [10 points] What are the possible values of the nullity (that is, the dimension of the kernel) of $T$? Justify your answer.

Solution: By the Rank-Nullity Theorem, the nullity of $T$ is equal to $\dim(V) - \text{rank}(T)$. Since the target vector space is one-dimensional, the rank is either 0 or 1. The dimension of $V$ is 3. Therefore, the nullity of $T$ is equal to either 2 or 3.

6. [20 points] Suppose that $\{u_1, u_2\}$ is a basis for a vector space $U$. Is the set

$$ \{u_1 - u_2, u_1 + 2u_2\} $$

also a basis for $U$? Justify your answer.

Solution: To decide if the set is linearly independent, suppose that

$$ a(u_1 - u_2) + b(u_1 + 2u_2) = 0. $$

Rearranging we have

$$ (a + b)u_1 + (-a + 2b)u_2 = 0. $$

Since $\{u_1, u_2\}$ is a basis, it is linearly independent and so

$$ a + b = 0, \quad -a + 2b = 0. $$

Adding these two equations, we get $3b = 0$ and so $b = 0$. Therefore, also $a = 0$. So the set

$$ \{u_1 - u_2, u_1 + 2u_2\} $$

is linearly independent.

We know that $\dim(U) = 2$ since $\{u_1, u_2\}$ is a basis. Therefore, the linearly independent set $\{u_1 - u_2, u_1 + 2u_2\}$ must also be a basis.
7. (a) [5 points] Explain what it means to say that a real number \( \lambda \) is an *eigenvalue* of an \( n \times n \) matrix \( A \).

**Solution:** It means that there is some \( x \in \mathbb{R}^n \) with \( x \neq 0 \) such that
\[
Ax = \lambda x.
\]

(b) [10 points] Calculate the eigenvalues of the matrix
\[
A = \begin{bmatrix}
2 & -1 & 0 \\
0 & 1 & 1 \\
0 & 3 & -1
\end{bmatrix}.
\]

**Solution:** The characteristic polynomial is
\[
\det \begin{bmatrix}
2 - \lambda & -1 & 0 \\
0 & 1 - \lambda & 1 \\
0 & 3 & -1 - \lambda
\end{bmatrix}
\]
which gives
\[
(2 - \lambda)((1 - \lambda)(-1 - \lambda) - 3) = (2 - \lambda)((\lambda^2 - 4) = -(\lambda - 2)^2(\lambda + 2).
\]
Therefore, the eigenvalues of \( A \) are \( \lambda = 2 \) (with multiplicity 2) and \( \lambda = -2 \) (with multiplicity 1).

(c) [15 points] Decide if \( A \) is diagonalizable or not. Justify your answer.

**Solution:** To find eigenvectors for \( \lambda = 2 \), we solve
\[
(A - 2I)x = 0
\]
which gives
\[
\begin{bmatrix}
0 & -1 & 0 \\
0 & -1 & 1 \\
0 & 3 & -3
\end{bmatrix} x = 0.
\]

In reduced row echelon form, this linear system is
\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} x = 0
\]
which has solutions
\[
x = \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \mid t \in \mathbb{R} \right\}.
\]
This eigenspace is 1-dimensional. Since the multiplicity of the eigenvalue 2 is 2, this means that \( A \) is **not** diagonalizable.
8. (a) [5 points] Let $U$ and $W$ be vector spaces. Explain what it means to say that a linear transformation $S : U \to W$ is invertible.

Solution: It means that there is some linear transformation $S^{-1} : W \to U$ such that $SS^{-1} = I_W$, the identity on $W$, and $S^{-1}S = I_U$, the identity on $U$. (Alternatively, one could also explain invertibility in terms of injectivity and surjectivity of $S$.)

(b) [5 points] Let $T : V \to V$ be a linear transformation, and let $\alpha = \{v_1, v_2\}$ be a basis for the vector space $V$. Suppose that the matrix of $T$ with respect to $\alpha$ is

\[
\begin{bmatrix}
3 & -1 \\
2 & 1
\end{bmatrix}
\]

Explain how you know that $T$ is invertible.

Solution: The matrix above has determinant $5 \neq 0$ so is an invertible matrix. Therefore $T$ is invertible.

(c) [10 points] Calculate $T^{-1}(v_1)$. Write your answer as a linear combination of the vectors $v_1$ and $v_2$.

Solution: The matrix for $T^{-1}$ is given by

\[
\begin{bmatrix}
3 & -1 \\
2 & 1
\end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}.
\]

Therefore

\[T^{-1}(v_1) = \frac{1}{5}v_1 - \frac{2}{5}v_2.\]