Amherst College
Department of Mathematics and Statistics

Comprehensive and Honors Qualifying Examination

◁ Multivariable Calculus and Linear Algebra ▷

January 2017

Number: _____

Senior: _____

Junior: _____

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space, and indicate whether you are a junior or a senior.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- The Multivariable Calculus and Linear Algebra Exam consists of Questions 1–8 that total to 200 points.

For Department Use Only:

Grader #1: _________________

Grader #2: _________________

1. (a) [15 points] Find an equation of form $ax+by+cz=d$ for the plane passing through $(-2,-1,4)$ that is perpendicular to the line with parametric equations $x=2t, y=3t-1, z=5-t$.

   (b) [15 points] Find an equation for the tangent plane to the sphere $x^2+y^2+z^2=3$ at the point $(1,-1,1)$.

2. Let $f(x,y)=\begin{cases} \dfrac{3x^3-5y^3}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$

   (a) [15 points] Compute $f_x(0,0)$ and $f_y(0,0)$.

   (b) [10 points] Is f is continuous at $(0,0)$? Justify your answer.

3. [20 points] Find the volume of the solid region bounded by the surfaces $y=0$, $y=3$, $z=x^2$, and $z=2-x^2$.

4. [25 points] Show that the line integral

$$\int_C z^2\,dx \;+\; 2y\,dy \;+\; 2xz\,dz$$

   depends only on the endpoints of the path C and not on the path taken between those endpoints.

5. Let V denote the vector space of polynomials of degree less than or equal to 2 with real coefficients. Let $T: V\to\mathbb{R}$ be a linear transformation.

   (a) [5 points] Explain what is meant by the *kernel*, or *null space*, of T.

   (b) [15 points] Prove that the kernel of T is a subspace of V.

   (c) [10 points] What are the possible values of the nullity (that is, the dimension of the kernel) of T? Justify your answer.

6. [20 points] Suppose that $\{\mathbf{u}_1,\mathbf{u}_2\}$ is a basis for a vector space U. Is the set

$$\{\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_1+2\mathbf{u}_2\}$$

   also a basis for U? Justify your answer.

7. (a) [5 points] Explain what it means to say that a real number λ is an *eigenvalue* of an $n\times n$ matrix A.

   (b) [10 points] Calculate the eigenvalues of the matrix $A=\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & -1 \end{bmatrix}$.

   (c) [15 points] Decide if the matrix A in part (b) is diagonalizable or not. Justify your answer.

8. (a) [5 points] Let U and W be vector spaces. Explain what it means to say that a linear transformation $S : U \to W$ is *invertible*.

   (b) [5 points] Let $T : V \to V$ be a linear transformation, and let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for the vector space V. Suppose that the matrix of T with respect to α is

$$\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

   Explain how you know that T is invertible.

   (c) [10 points] Calculate $T^{-1}(\mathbf{v}_1)$. Write your answer as a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.