Comprehensive and Honors Qualifying Examination

Multivariable Calculus and Linear Algebra

January 2017

Number: _____
Senior: _____
Junior: _____

Read This First:

• This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.

• Write your number (not your name) in the above space, and indicate whether you are a junior or a senior.

• For any given problem, you may use the back of the previous page for scratch work. Put your final answers in the spaces provided.

• Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.

• In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

• The Multivariable Calculus and Linear Algebra Exam consists of Questions 1–8 that total to 200 points.

For Department Use Only:

Grader #1: ________________
Grader #2: ________________
1. (a) [15 points] Find an equation of form $ax + by + cz = d$ for the plane passing through $(-2, -1, 4)$ that is perpendicular to the line with parametric equations $x = 2t, y = 3t - 1, z = 5 - t$.

(b) [15 points] Find an equation for the tangent plane to the sphere $x^2 + y^2 + z^2 = 3$ at the point $(1, -1, 1)$.

2. Let $f(x, y) = \begin{cases} \frac{3x^3 - 5y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

(a) [15 points] Compute $f_x(0, 0)$ and $f_y(0, 0)$.

(b) [10 points] Is $f$ is continuous at $(0, 0)$? Justify your answer.

3. [20 points] Find the volume of the solid region bounded by the surfaces $y = 0$, $y = 3$, $z = x^2$, and $z = 2 - x^2$.

4. [25 points] Show that the line integral

$$ \int_C z^2 \, dx + 2y \, dy + 2xz \, dz $$

depends only on the endpoints of the path $C$ and not on the path taken between those endpoints.

5. Let $V$ denote the vector space of polynomials of degree less than or equal to 2 with real coefficients. Let $T : V \to \mathbb{R}$ be a linear transformation.

(a) [5 points] Explain what is meant by the kernel, or null space, of $T$.

(b) [15 points] Prove that the kernel of $T$ is a subspace of $V$.

(c) [10 points] What are the possible values of the nullity (that is, the dimension of the kernel) of $T$? Justify your answer.

6. [20 points] Suppose that $\{u_1, u_2\}$ is a basis for a vector space $U$. Is the set

$$ \{u_1 - u_2, u_1 + 2u_2\} $$

also a basis for $U$? Justify your answer.

7. (a) [5 points] Explain what it means to say that a real number $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$.

(b) [10 points] Calculate the eigenvalues of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & -1 \end{bmatrix}$.

(c) [15 points] Decide if the matrix $A$ in part (b) is diagonalizable or not. Justify your answer.
8. (a) [5 points] Let $U$ and $W$ be vector spaces. Explain what it means to say that a linear transformation $S : U \to W$ is invertible.

(b) [5 points] Let $T : V \to V$ be a linear transformation, and let $\alpha = \{v_1, v_2\}$ be a basis for the vector space $V$. Suppose that the matrix of $T$ with respect to $\alpha$ is

\[
\begin{bmatrix}
3 & -1 \\
2 & 1
\end{bmatrix}
\]

Explain how you know that $T$ is invertible.

(c) [10 points] Calculate $T^{-1}(v_1)$. Write your answer as a linear combination of the vectors $v_1$ and $v_2$. 