Solutions to the Algebra problems on the Comprehensive Examination of January 27, 2017

1. [25 points] Let G be a group, let $H \subseteq G$ be a subgroup, and let $N \subseteq G$ be a **normal** subgroup. Define
$$NH = \{nh : n \in N, h \in H\}.$$
Prove that NH is a subgroup of G.

   Solution: (Nonempty): Since N and H are both nonempty, we may choose $x \in N$ and $h \in H$. Then $xh \in NH$, so $NH \neq \varnothing$.

   (Closed under $*$): Given $x, y \in NH$, write $x = n_1h_1$ and $y = n_2h_2$, for some $n_1, n_2 \in N$ and $h_1, h_2 \in H$. Since N is a normal subgroup of G, we have $h_1N = Nh_1$. Thus, there is some $n_3 \in N$ such that $h_1n_2 = n_3h_1$. So
$$xy = (n_1h_1)(n_2h_2) = n_1(h_1n_2)h_2 = n_1(n_3h_1)h_2 = (n_1n_3)(h_1h_2) \in NH,$$
as desired, since $n_1n_3 \in N$ and $h_1h_2 \in H$.

   (Closed under inverses): Given $x \in NH$, write $x = nh$ with $n \in N$ and $h \in H$. Since N is a normal subgroup of G, we have $hN = Nh$. Thus, there is some $n_1 \in N$ such that $nh = hn_1$. So
$$x^{-1} = (nh)^{-1} = (hn_1)^{-1} = n_1^{-1}h^{-1} \in NH,$$
as desired, since $n_1^{-1} \in N$ and $h^{-1} \in H$.

2. [17 points] Let G be a finite group, and suppose that there is an element $a \in G$ with the property that $a^2 = a^{-1}$ but a is not the identity. Prove that the order of G is divisible by 3.

   Solution: Since $a^2 = a^{-1}$, we have that $a(a^2) = aa^{-1}$, or equivalently, $a^3 = e$, where e is the identity element in G. Then $o(a)|3$, and hence $o(a)$ is either 1 or 3. Since $a \neq e$, we have $o(a) \neq 1$, so the order of a must be 3. By (a corollary of) Lagrange's Theorem, $o(a)$ divides the order of G. That is, the order of G is divisible by 3, as desired.

3. Consider the group S_8 of permutations of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

   (a) [10 points] Find an element of S_8 of order 15.
   (Don't forget to explain or prove why it has order 15.)

   Solution: Let $\sigma = (1, 2, 3)(4, 5, 6, 7, 8) \in S_8$. Since we have already written σ as a product of disjoint cycles, we have $o(\sigma) = \text{lcm}(3, 5) = 15$.

   (b) [15 points] Prove that there are no **odd** permutations in S_8 of order 15.

   Solution: Suppose $\sigma \in S_8$ satisfies $|\sigma| = 15$. Write $\sigma = \alpha_1\alpha_2 \cdots \alpha_k$ as a product of k **disjoint** cycles $\alpha_1, \dots, \alpha_k$ for some positive integer k. Because these cycles are disjoint, we have
$$15 = o(\sigma) = \text{lcm}(o(\alpha_1), o(\alpha_2), \dots, o(\alpha_k)).$$

By definition of the least common multiple, each $o(\alpha_j)$ divides 15. Thus, each $o(\alpha_j) \in \{3, 5, 15\}$, since these are the only divisors of 15 which are greater than 1. Hence, each α_j is a cycle of odd length. We know that any **cycle** of odd length is an even permutation. Thus, each α_j is even. Therefore, σ is also even, because it is the product of even permutations.

4. Let R be a ring.

   (a) [8 points] Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.
   If you use any other technical terms like "closed," "subring," "group," "subgroup," etc., you must fully define those terms as well.

   Solution: $I \subseteq R$ is an ideal if:
   1. $I \neq \varnothing$.
   2. For all $x, y \in I$, we have $x - y \in I$
   3. For all $x \in I$ and $r \in R$, we have $rx \in I$ and $xr \in I$.

   (b) [25 points] Suppose that R is commutative and has a multiplicative identity 1. Let $I \subseteq J \subseteq R$ be ideals, and suppose that the quotient ring R/I is a field.

   If $I \subsetneq J$, prove that $1 \in J$.

   (In fact, it is a Theorem from Math 350 that $J = R$ in this case, but you are only being asked to prove that $1 \in J$. In particular, however, you may **not** quote the $J = R$ theorem.)

   Solution: Because $I \subsetneq J$, there exists $r \in J \smallsetminus I$. By the coset relation, $r \notin I$ implies $I + r \neq I + 0 = 0_{R/I}$.
   Thus, $I + r$ is nonzero. By the definition of field, every nonzero element of R/I has a multiplicative inverse. Hence, there exists $I + s \in R/I$ such that

   $$(I + r)(I + s) = I + 1.$$

   So $I + rs = I + 1$, and therefore $1 - rs \in I \subseteq J$ by the coset relation.
   Note that $rs \in J$, because J is an ideal of R and $r \in J$. Thus, since J is closed under addition, we have $1 = (1 - rs) + rs \in J$.