Amherst College Department of Mathematics and Statistics

Solutions to the Algebra problems on the Comprehensive Examination of January 26, 2018

1. **(25 points)** Let G_1 and G_2 be groups, let $H_1 \subseteq G_1$ be a subgroup, and let $\phi : G_1 \to G_2$ be a homomorphism. The set

$$H_2 = \{\phi(x) : x \in H_1\}$$

is called the *image of* H_1 *under* ϕ, sometimes denoted $\phi(H_1)$.
Prove that H_2 is a subgroup of G_2.

[This is a standard theorem in Math 350. You must actually prove it, not just quote it.]

Solution: It suffices to show that H_2 is nonempty, closed under $\cdot$, and closed under $^{-1}$.
Nonempty: Let e_1 be the identity in G_1. Then e_1 is also the identity in H_1 (e.g. $e_1 \in H_1$), so that $\phi(e_1) \in H_2$, hence H_2 is nonempty. (In fact, because ϕ is a homomorphism, $\phi(e_1) = e_2$, where e_2 is the identity in G_2.)

Closed under $\cdot$: Let $z, w \in H_2$. Then there exists $x, y \in H_1$ such that $z = \phi(x), w = \phi(y)$. Since H_1 is a subgroup, $xy \in H_1$, hence $\phi(xy) \in H_2$. But ϕ is a homomorphism, so that $\phi(xy) = \phi(x)\phi(y)$. Hence, $zw = \phi(x)\phi(y) \in H_2$, so that H_2 is closed under $\cdot$.

Closed under $^{-1}$: Let $z \in H_2$. Then there exists $x \in H_1$ such that $z = \phi(x)$. Since H_1 is a subgroup, $x^{-1} \in H_1$, hence $\phi(x^{-1}) \in H_2$. But ϕ is a homomorphism, so that $\phi(x^{-1}) = \phi(x)^{-1}$. Hence, $z^{-1} = \phi(x)^{-1} \in H_2$, so that H_2 is closed under $^{-1}$.

2. **(25 points)** Let G be a group, let $N \subseteq G$ be a normal subgroup, and let $m \geq 1$ be an integer. Suppose that for every element $y \in G/N$, the order of y divides m.
Prove that for all $x \in G$, we have $x^m \in N$.

Solution: Let $x \in G$. Then $xN \in G/N$. Let n be the order of xN in G/N. By hypotheses, n divides m, so there is some integer k such that $nk = m$. Then $(xN)^n = N$ (the identity in G/N), so that $x^m N = (xN)^m = (xN)^{nk} = ((xN)^n)^k = N^k = N$. That is, $x^m N = N$, which implies $x^m \in N$ as claimed.

3. **(25 points)** Consider the group S_8 of permutations of the set $\{1, 2, 3, \ldots, 8\}$. Let $\sigma, \tau \in S_8$ be the permutations

$$\sigma = (1, 2, 3)(4, 5, 6) \quad \text{and} \quad \tau = (3, 5)(1, 7, 8, 4).$$

(a) **(7 points)** Write $\sigma^2\tau$ as a product of **disjoint** cycles.

Solution: $\sigma^2\tau = (1, 2, 3)(4, 5, 6)(1, 2, 3)(4, 5, 6)(3, 5)(1, 7, 8, 4) = (1, 7, 8, 6, 5, 2)(3, 4)$

(b) **(9 points)** Compute the **order** of each of σ, τ, and $\sigma^2\tau$.

Solution: σ is given as the product of two disjoint 3-cycles, so the order of σ is $\text{lcm}(3, 3) = 3$. Similarly, τ is given as the product of a 2-cycle and a 4-cycle which are disjoint, so the order of σ is $\text{lcm}(2, 4) = 4$. Finally, from part (a), $\sigma^2\tau$ is the product of a 6-cycle and a 2-cycle which are disjoint, so the order of $\sigma^2\tau$ is $\text{lcm}(6, 2) = 6$.

(c) **(9 points)** Decide whether each of σ, τ, and $\sigma^2\tau$ is an **even** or **odd** permutation; don't forget to justify.

Solution: σ is a product of two 3-cycles (which are both even) so is even + even = even.
τ is a product of a 2-cycle (odd) and a 4-cycle (odd), so is odd + odd = even.
$\sigma^2\tau$ is a product of a 6-cycle (odd) and a 2-cycle (odd), so is odd + odd = even.

4. **(25 points)** Let R be a ring.

(a) **(6 points)** Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.
If you use any other technical terms like "closed," "subring," "group," "subgroup," etc., you must fully define those terms as well.

Solution: A subset $I \subseteq R$ is an ideal of R if

- $I \neq \emptyset$,
- $x, y \in I$ implies $x - y \in I$,
- $x \in I$ and $r \in R$ implies $xr \in I$ and $rx \in I$.

(b) **(19 points)** Let $I, J \subseteq R$ be ideals, and define

$$I + J = \{x + y : x \in I \text{ and } y \in J\}.$$

Prove that $I + J$ is an ideal of R.

Solution: We will show $I + J$ satisfies the three conditions given in part (a).

- Since I and J are ideals, they are non-empty. In particular, $0_R \in I$ and $0_R \in J$ (where 0_R is the zero element in R). Then $0_R + 0_R = 0_R \in I + J$. Hence, $I + J \neq \emptyset$.
- Let $z_1, z_2 \in I + J$. Then there exist $x_1, x_2 \in I$ and $y_1, y_2 \in J$ such that $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$. Since I is an ideal, $x_1 - x_2 \in I$. Similarly, since J is an ideal, $y_1 - y_2 \in J$. Thus,
$$z_1 - z_2 = (x_1 + y_1) - (x_2 + y_2) = (x_1 - x_2) + (y_1 - y_2) \in I + J.$$
- Let $r \in R$ and $z \in I + J$. Then there exists $x \in I, y \in J$ such that $z = x + y$. Since I is an ideal, rx and xr are in I. Similarly, since J is an ideal, ry and yr are in J. Thus, $rz = r(x + y) = rx + ry \in I + J$, and $zr = (x + y)r = xr + yr \in I + J$.