Amherst College
Department of Mathematics and Statistics

Comprehensive Examination

◁ Algebra ▷

January 2018

Number: _____

Read This First:

- This is a closed-book examination. No books, notes, cell phones, electronic devices of any sort, or other aids are permitted. Cell phones are to be silenced and out of sight.
- Write your number (***not*** your name) in the above space.
- For any given problem, you may use the back of the ***previous*** page for scratch work. Put your final answers in the spaces provided.
- Additional sheets of paper will be available if you need them. If you use an additional sheet, label it carefully and be sure to include your number.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable. Show all your work, and justify your answers.
- The Algebra Exam consists of Questions 1–4 that total to 100 points.

For Department Use Only:

Grader #1: _________________

Grader #2: _________________

1. [25 points] Let G_1 and G_2 be groups, let $H_1 \subseteq G_1$ be a subgroup, and let $\phi : G_1 \to G_2$ be a homomorphism. The set

$$H_2 = \{\phi(x) : x \in H_1\}$$

is called the *image of H_1 under ϕ*, sometimes denoted $\phi(H_1)$.

Prove that H_2 is a subgroup of G_2.

[This is a standard theorem in Math 350. You must actually prove it, not just quote it.]

2. [25 points] Let G be a group, let $N \subseteq G$ be a normal subgroup, and let $m \geq 1$ be an integer. Suppose that for every element $y \in G/N$, the order of y divides m.

Prove that for all $x \in G$, we have $x^m \in N$.

3. [25 points] Consider the group S_8 of permutations of the set $\{1, 2, 3, \ldots, 8\}$. Let $\sigma, \tau \in S_8$ be the permutations

$$\sigma = (1, 2, 3)(4, 5, 6) \quad \text{and} \quad \tau = (3, 5)(1, 7, 8, 4).$$

(a) [7 points] Write $\sigma^2\tau$ as a product of **disjoint** cycles.

(b) [9 points] Compute the **order** of each of σ, τ, and $\sigma^2\tau$.

(c) [9 points] Decide whether each of σ, τ, and $\sigma^2\tau$ is an **even** or **odd** permutation; don't forget to justify.

4. [25 points] Let R be a ring.

(a) [6 points] Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.

If you use any other technical terms like "closed," "subring," "group," "subgroup," etc., you must fully define those terms as well.

(b) [19 points] Let $I, J \subseteq R$ be ideals, and define

$$I + J = \{x + y : x \in I \text{ and } y \in J\}.$$

Prove that $I + J$ is an ideal of R.